PART IIA 2005
3E4: Modelling choice
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Datasheet: None

Engineering Tripos Part IIA – 2005

Module 3E4 – Modelling Choice

Solution to the Exam Paper

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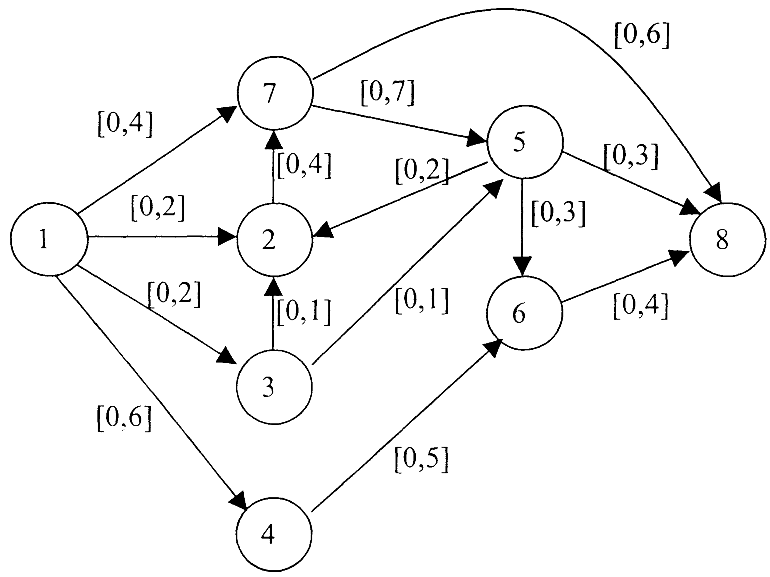
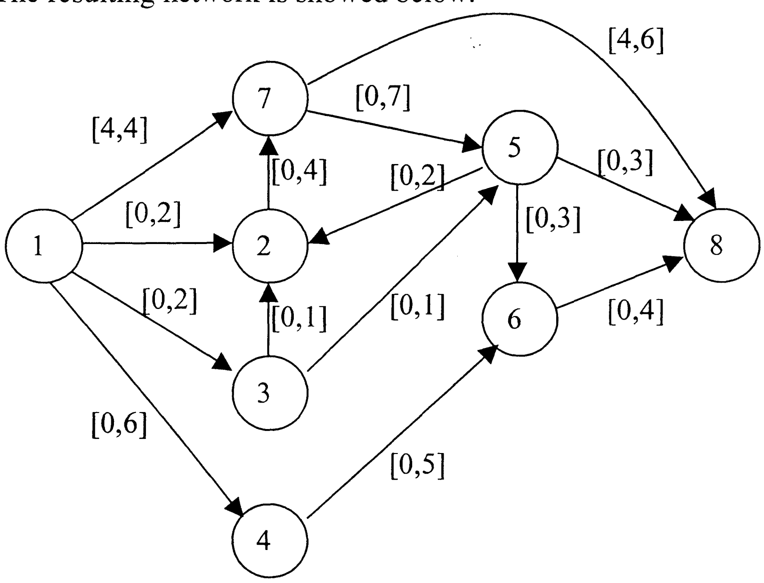
1.

(a)

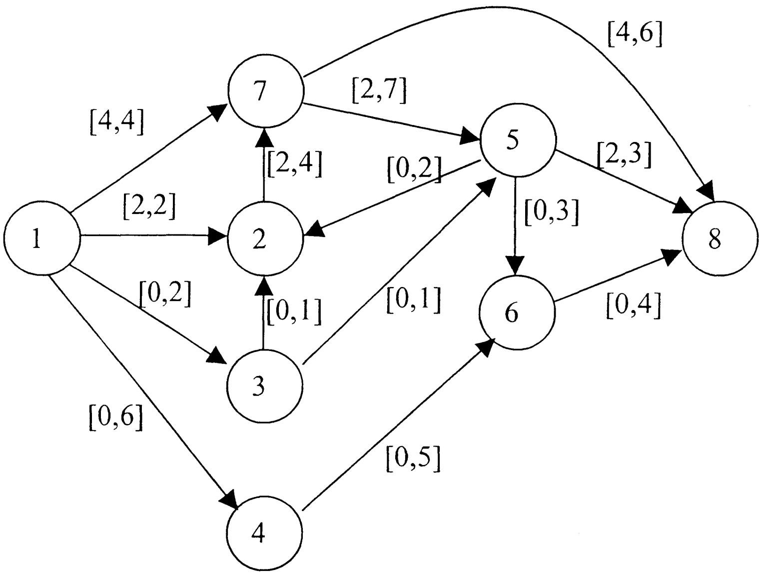


(i) We start the Ford-Fulkerson algorithm using the current feasible flow as a starting solution and apply the labelling algorithm to find an augmenting path, if any. We put the source node into the list, L:= {1}. Then extract 1 from the list and try to both forward labels to all the unlabeled nodes adjacent to 1 by unsaturated arcs, and to backward labels to all the unlabeled nodes adjacent to 1 by arcs that have positive flows. Therefore $\lambda(2)=(1,2)$, $\lambda(3)=(1,2)$, $\lambda(4)=(1,6)$, $\lambda(7)=(1,4)$, and we put 2, 3, 4 and 7 into the list L:= {2,3,4,7}. Extract 7, and apply the labelling procedure obtaining $\lambda(5)=(7,4)$, $\lambda(8)=(7,4)$, L:={2,3,6,8} and stop, since 8 has been labelled. An augmenting path has been found and the current flow is not optimal. Build now the augmenting path backtracking on the label first-components from node 8, that is 8 – 7 – 1. Send along such path the flow amount given by the second-component of $\lambda(8)$, that is increase of 4 units the flow along the augmenting path. The resulting network is showed below.

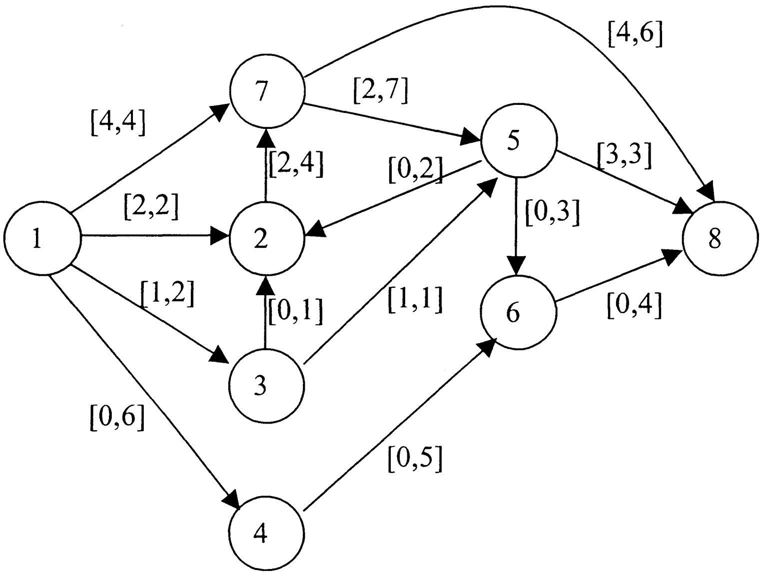


With respect to the new flow configuration, now we look for an augmenting path by inspecting the network.

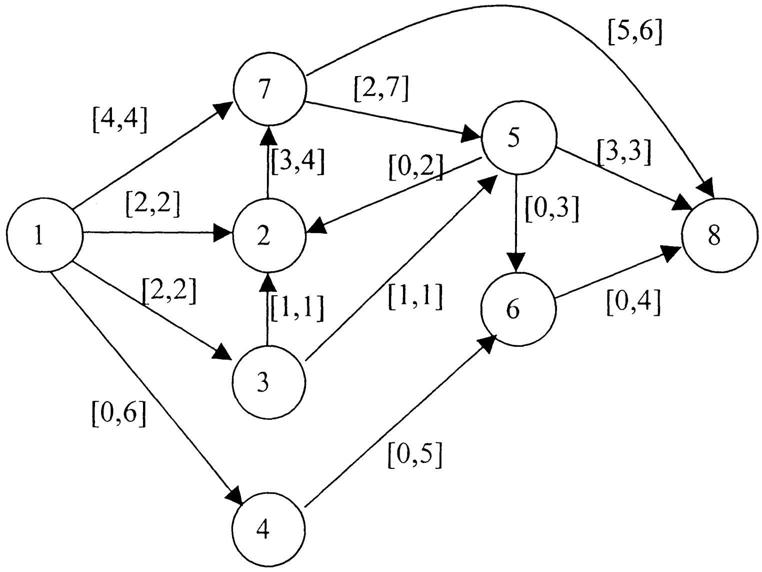
Notice that an augmenting path is 1 – 2 – 7 – 5 – 8. The forward arcs are {(1,2),(2,7),(7,5),(5,8)}, and the maximum flow increase is given by min{2,4,2,3}=2. No backward arcs are available. Therefore send along such path two units of flow and obtain the following network.



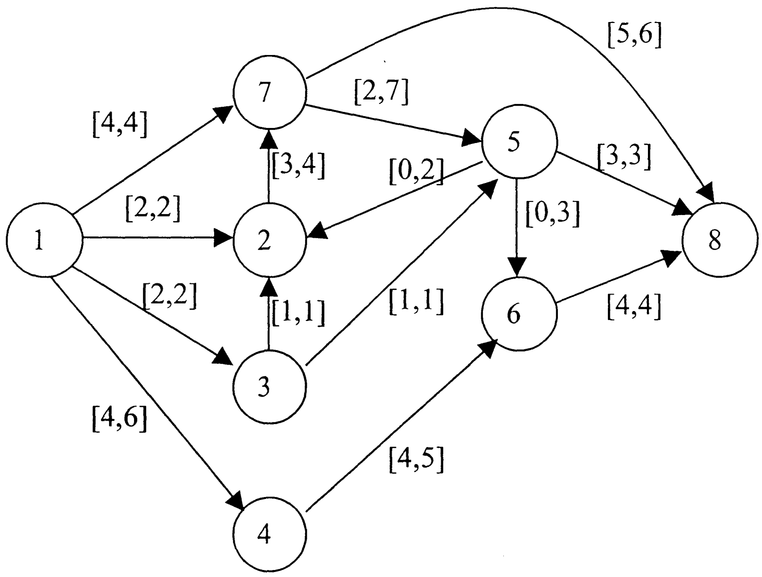
With respect to the new flow configuration, now we look for an augmenting path by inspecting the network.
Notice that an augmenting path is 1 – 3 – 5 – 8. The forward arcs are {(1,3),(3,5),(5,8)}, and the maximum flow increase is given by min{2,1,1}=1. No backward arcs are available. Therefore send along such path one unit of flow and obtain the following network.



Notice that an augmenting path is 1 – 3 – 2 – 7 – 8. The forward arcs are {(1,3),(3,2),(2,7) ,(7,8)}, and the maximum flow increase is given by min{1,1,2,2}=1. No backward arcs are available. Therefore send along such path one unit of flow and obtain the following network.

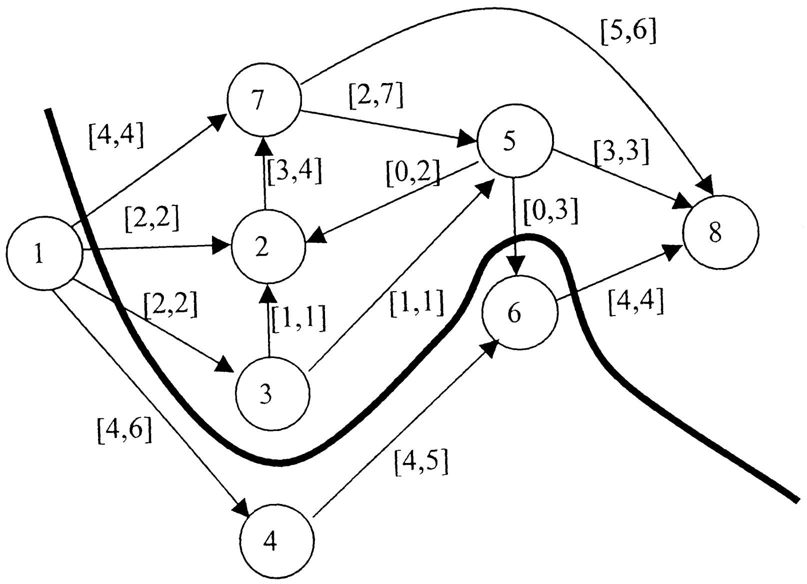


Notice that an augmenting path is 1 – 4 – 6 – 8. The forward arcs are {(1,4),(4,6) ,(6,8)}, and the maximum flow increase is given by min{6,5,4}=4. No backward arcs are available. Therefore send along such path four units of flow and obtain the following network.



It is easy to see that such network does not contain any augmenting path, therefore the algorithm stops as the current flow distribution is optimal, and the value of the flow is 12.

(ii) It is sufficient, according to the "Max Flow – Min Cut" theorem, to find an s-t cut (where s=1 and t=8) on the network, whose capacity equals the value v=12 of the current flow distribution. In fact the picture below shows that the cut U={1,4,6} and W={2,3,5,7,8} satisfies the "Max Flow – Min Cut" theorem.

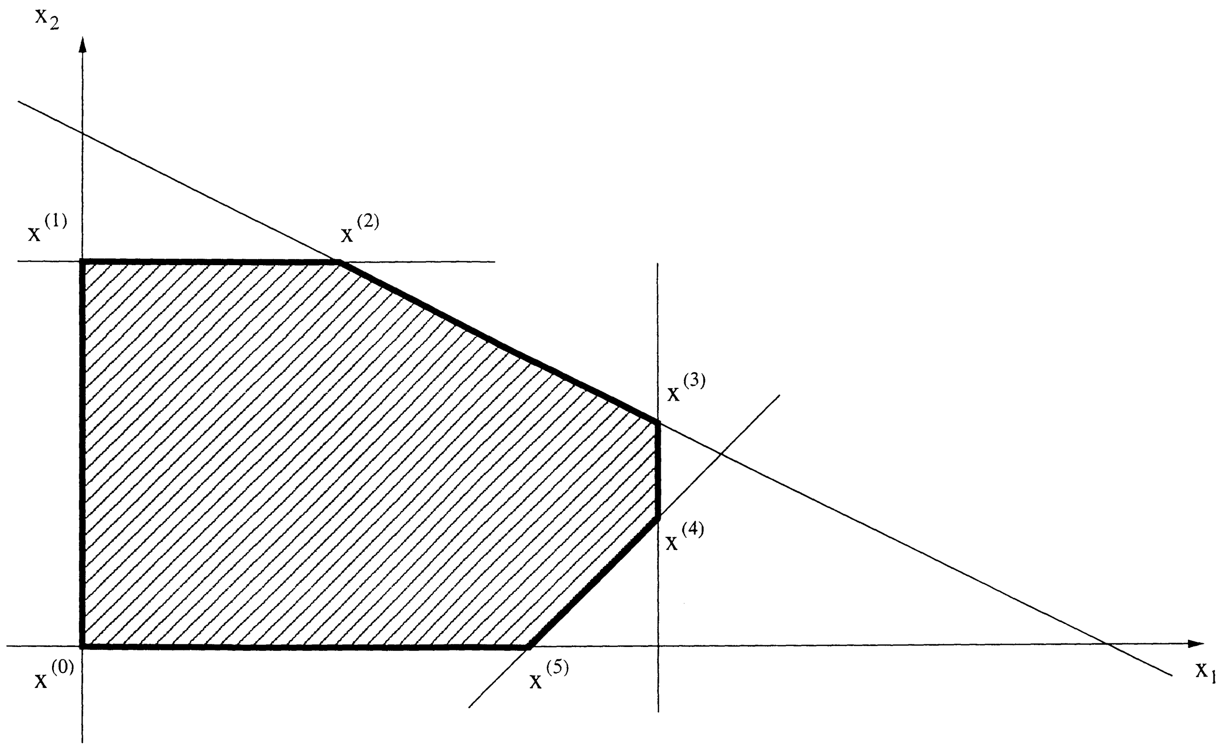


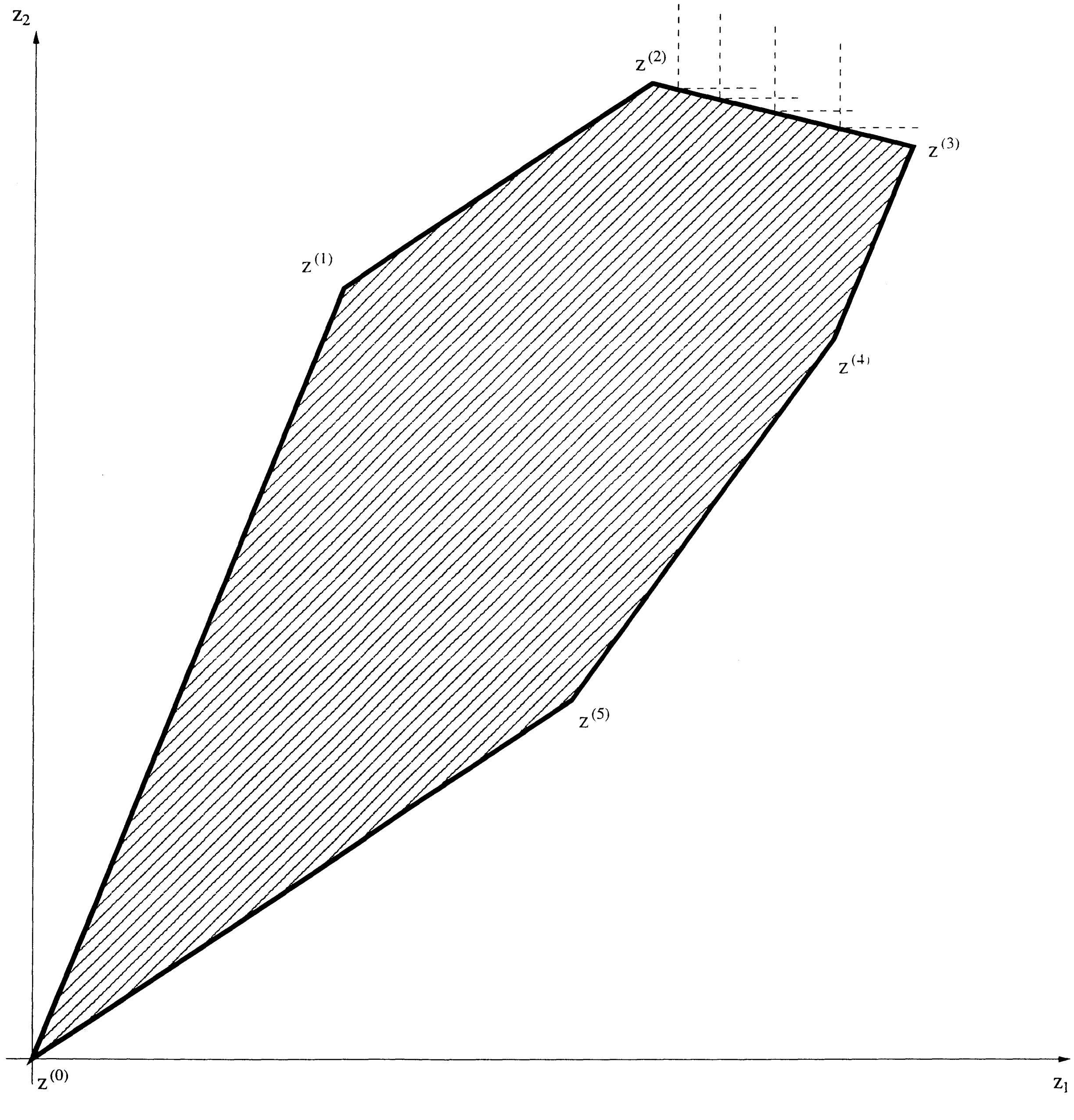
[5,6]
7
[2,7]
[4,4]
5
[3,3]
[3,4]
[0,2]
[2,2]
[0,3]
8
1
2
[2,2]
[1,1]
[1,1]
6
[4,4]
3
[4,6]
[4,5]
4

(b)

$$\begin{aligned} &\text{Max} && 3x_1 + 2x_2 \\ &\text{Max} && 2x_1 + 5x_2 \\ &\text{subject to} && x_1 - x_2 \leq 7 \\ &&& x_1 + 2x_2 \leq 16 \\ &&& x_1 \leq 9 \\ &&& x_2 \leq 6 \\ &&& x_1, x_2 \geq 0 \end{aligned}$$

(i) In the following both the feasible region and the objective space are reported:





where

$$x^{(0)}=(0,0),\ x^{(1)}=(0,6),\ x^{(2)}=(4,6),\ x^{(3)}=(9,7/2),\ x^{(4)}=(9,2),\ x^{(5)}=(7,0),$$

and

$$z^{(0)}=(0,0),\ z^{(1)}=(12,30),\ z^{(2)}=(24,38),\ z^{(3)}=(34,71/2),\ z^{(4)}=(31,28),\ z^{(5)}=(21,14).$$

(ii) By simply checking the picture of the objective space, the values of the Pareto solutions are obtained along the segment joining $z^{(2)}$ and $z^{(3)}$. Therefore the Pareto solutions are all the points lying along the segment joining $x^{(2)}$ and $x^{(3)}$, see the feasible region.

2.

(a) First we define the variables of the problem.
Let x_{ij} , for i=1,2 and j=1,2,3, denote the amount of tyre **Rain** produced on machine i during the month j, where we assume that i=1 denotes machine **M1**, and i=2 denotes machine **M2**, while j=1,2,3 denote months **May**, **June** and **July**, respectively. Similarly we define y_{ij} , for i=1,2 and j=1,2,3, as the amount of tyre **Slick** produced on machine i during the month j.
Moreover, let x_i , for i=1,2, denote the amount of tyre **Rain** on stock at the end of month i, and y_i, for i=1,2, the amount of tyre **Slick** on stock at the end of month i.
Now we consider the objective function of the problem. It is requested to minimize the production cost, taking into account possible inventory costs arising. Therefore the following cost function appears the most appropriate:

Minimize

$$1.2*(x_{11} + x_{12} + x_{13}) + 1{,}44*(x_{21} + x_{22} + x_{23}) + 2.16*(y_{11} + y_{12} + y_{13}) + 1{,}8*(y_{21} + y_{22} + y_{23}) +$$
$$4*(x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23}) + 5*(y_{11} + y_{12} + y_{13} + y_{21} + y_{22} + y_{23}) +$$
$$0.50*(x_1 + x_2 + y_1 + y_2)$$

In fact the first row contains contribution to the cost of machine time utilization, the second row the raw material costs, and the third row the inventory costs.
As for the constraints, we first focus on limited availability of machines. In particular for the machine **M1** we obtain

$$0.10* x_{11} + 0.18* y_{11} \leq 500 \text{ (May)}$$
$$0.10* x_{12} + 0.18* y_{12} \leq 650 \text{ (June)}$$
$$0.10* x_{13} + 0.18* y_{13} \leq 300 \text{ (July)}$$

while for the machine **M2** we obtain:

$$0.12* x_{21} + 0.15* y_{21} \leq 600 \text{ (May)}$$
$$0.12* x_{22} + 0.15* y_{22} \leq 700 \text{ (June)}$$
$$0.12* x_{23} + 0.15* y_{23} \leq 450 \text{ (July).}$$

Now we consider the constraints regarding requests and possible production storage. Then, for instance, the overall amount of tyres **Rain** produced on both machines in **May** should equalize the amount of **Rain** tyre requested (4,500) plus the possible exceeding production. Namely,

$$x_{11} + x_{21} = 4500 + x_1 \text{ (May).}$$

Similarly, but taking into account possible nonempty inventory at the end of May and/or June, we obtain:

$$x_{12} + x_{22} + x_1 = 5000 + x_2 \text{ (June)}$$
$$x_{13} + x_{23} + x_2 = 2000 \qquad \text{(July).}$$

Finally, the same constraints as above hold for the **Slick** tyres:

$y_{11} + y_{21} = 3000 + y_1$ (May)
$y_{12} + y_{22} + y_1 = 3500 + y_2$ (June)
$y_{13} + y_{23} + y_2 = 8000$ (July).

The whole model is presented below, where the obvious nonnegativity constraints on all the variables have been introduced too.

Minimize

$1.2*(x_{11} + x_{12} + x_{13}) + 1{,}44*(x_{21} + x_{22} + x_{23}) + 2.16*(y_{11} + y_{12} + y_{13}) + 1{,}8*(y_{21} + y_{22} + y_{23}) +$
$4*(x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23}) + 5*(y_{11} + y_{12} + y_{13} + y_{21} + y_{22} + y_{23}) +$
$0.50*(x_1 + x_2 + y_1 + y_2)$

Subject to:

$0.10* x_{11} + 0.18* y_{11} \leq 500$
$0.10* x_{12} + 0.18* y_{12} \leq 650$
$0.10* x_{13} + 0.18* y_{13} \leq 300$

$0.12* x_{21} + 0.15* y_{21} \leq 600$
$0.12* x_{22} + 0.15* y_{22} \leq 700$
$0.12* x_{23} + 0.15* y_{23} \leq 450$

$x_{11} + x_{21} = 4500 + x_1$
$x_{12} + x_{22} + x_1 = 5000 + x_2$
$x_{13} + x_{23} + x_2 = 2000$

$y_{11} + y_{21} = 3000 + y_1$
$y_{12} + y_{22} + y_1 = 3500 + y_2$
$y_{13} + y_{23} + y_2 = 8000$

$x_{ij} \geq 0$, i=1,2; j=1,2,3;
$y_{ij} \geq 0$, i=1,2; j=1,2,3;
$x_i \geq 0$, i=1,2;
$y_i \geq 0$, i=1,2.

(b) Max $-2x_1 - x_2 - 3x_3 + 2x_4$

subject to

$$3x_1 + 2x_2 - 4x_4 - x_5 = 1$$
$$x_1 + 2x_3 + 2x_4 + x_5 = 3$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

(i) We first verify that the point (1,0,0,0,2) is feasible by just plugging it into the two equality constraints, and observing that the nonnegativity constraints are also satisfied. Notice that there are two equalities constraints and 5 variables, then the 2×5 constraint matrix A is the following:

A=

3	2	0	4	−1
1	0	2	2	1

Therefore, in order to check that the feasible point (1,0,0,0,2) is also a basic solution, after observing that it has exactly 2 nonzero components, as would be expected, it suffices to check that column 1 and 5 in the matrix, corresponding to the nonzeros, are in fact linearly independent. This is actually the case since

$$\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} = 4$$

As for possible optimality of this basic feasible solution, we observe that this is not the case. In fact, given

$$B = \begin{matrix} 3 & -1 \\ 1 & 1 \end{matrix} \qquad N = \begin{matrix} 2 & 0 & 4 \\ 0 & 2 & 2 \end{matrix} \qquad B^{-1} = \begin{matrix} 1/4 & 1/4 \\ -1/4 & 3/4 \end{matrix}$$

$p=(-2, -1, -3,2,0)$, $p_B=(-2,0)$, $p_N=(-1,-3,2)$

we obtain the following reduced cost vector $r^T = (p_N)^T - (p_B)^T B^{-1} N =(0, -2, 5)$. Since there exists one positive reduced cost, the basic feasible solution is not optimal.

(ii) Using the basic feasible solution (1,0,0,0,5) as a starting BFS, we first notice that the current simplex tableau with respect to the basis made up of columns 1 and 5 in the matrix A is the following:

$$x_1 = 1 - 1/2x_2 - 1/2x_3 - 3/2x_4$$
$$x_5 = 2 + 1/2x_2 - 3/2x_3 - 1/2x_4$$

where $x_B = B^{-1}b - B^{-1}Nx_N$. Since the reduced cost corresponding to x_4 is positive, then it is convenient to increase the x_4-value by letting this variable enter the basis. To understand which variable should leave the current basis it suffices to solve the following system

$$1 - 3/2x_4 >= 0$$
$$2 - 1/2x_4 >= 0$$

Finally, notice that for any $\alpha \in [0,1]$ it holds that $\alpha c^T x^* + (1-\alpha) c^T x^\# = z$, hence that $c^T y(\alpha) = z$ as well. Therefore for any $\alpha \in [0,1]$, $y(\alpha)$ is an optimal solution of the LP, and the thesis is proved.

3.

(a) Consider the following integer linear program (ILP)

[P_0]

Min $z = 4x + y$

subject to $3x + y \geq 3$

$4x + 3y \geq 6$

$x + 2y \leq 4$

$x, y \geq 0$, integer

(i) The linear relaxed of the master program, first note of the branch and bound tree, is

[LP_0]

Min $z = 4x + y$

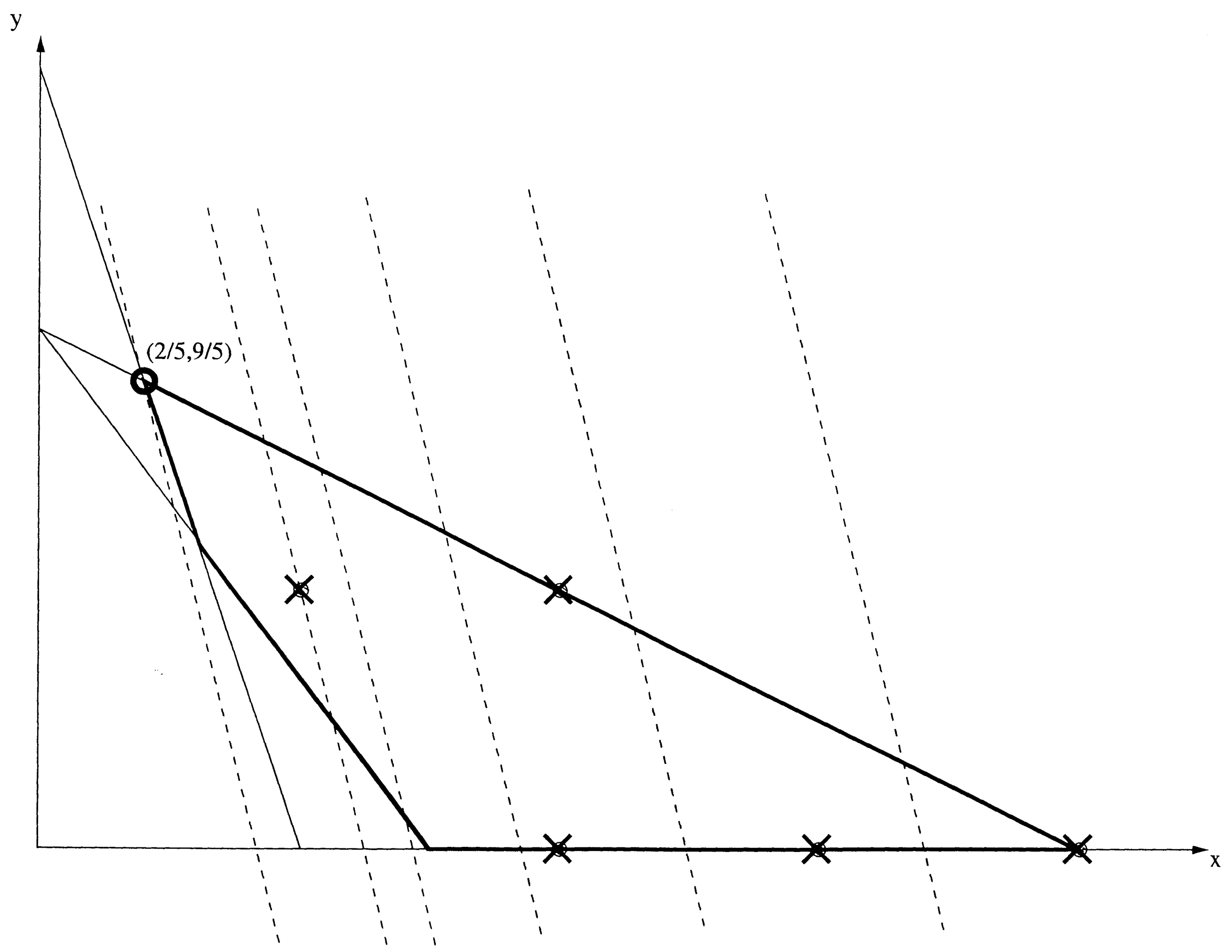
subject to $3x + y \geq 3$

$4x + 3y \geq 6$

$x + 2y \leq 4$

$x, y \geq 0$

Its solution can be found graphically at the point (2/5,9/5), whose value is L_0= 17/5 as shown in the picture below.



The solution is not integer, therefore we branch on the x-variable and two new ILPs are added to the B&B tree, namely:

[P_1]

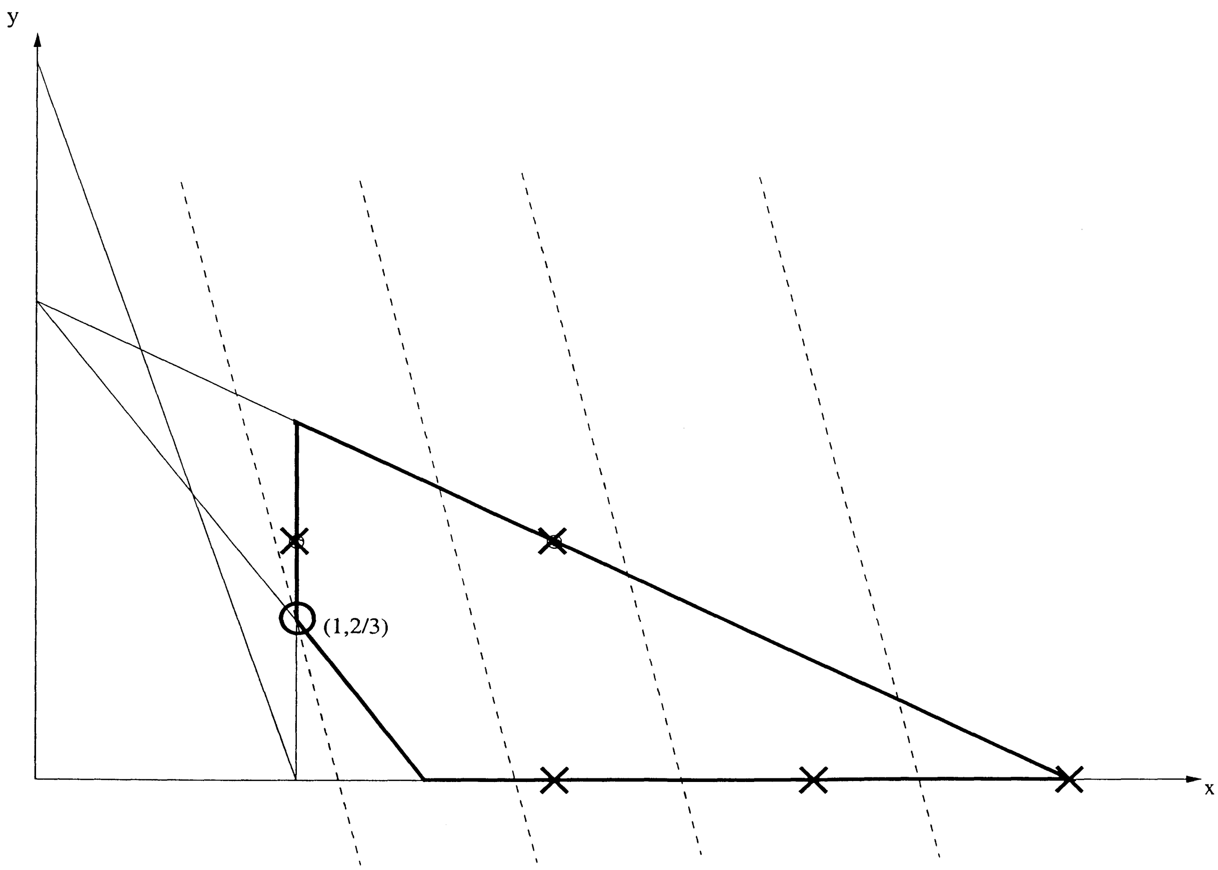
$$\begin{aligned} \text{Min} \quad & z = 4x + y \\ \text{subject to } & 3x + y \geq 3 \\ & 4x + 3y \geq 6 \\ & x + 2y \leq 4 \\ & x \leq 0 \\ & x, y \geq 0, \text{ integer} \end{aligned}$$

and

[P_2]

$$\begin{aligned} \text{Min} \quad & z = 4x + y \\ \text{subject to } & 3x + y \geq 3 \\ & 4x + 3y \geq 6 \\ & x + 2y \leq 4 \\ & x \geq 1 \\ & x, y \geq 0, \text{ integer} \end{aligned}$$

Extract now the linear relaxation of problem [P_1]. By just looking at the picture above, it can be stated infeasibility of such program. Therefore the [P_1] branch can be pruned. Extract then [P_2], whose graphical representation follows



The relaxed program solution can be found graphically at the point (1,2/3), whose value is L_2 = 14/3. Since the solution is not integer, we have to branch once again. In particular, this time we can only branch with respect to the y-variable, and hence we obtain the following two ILPs

[P_3]

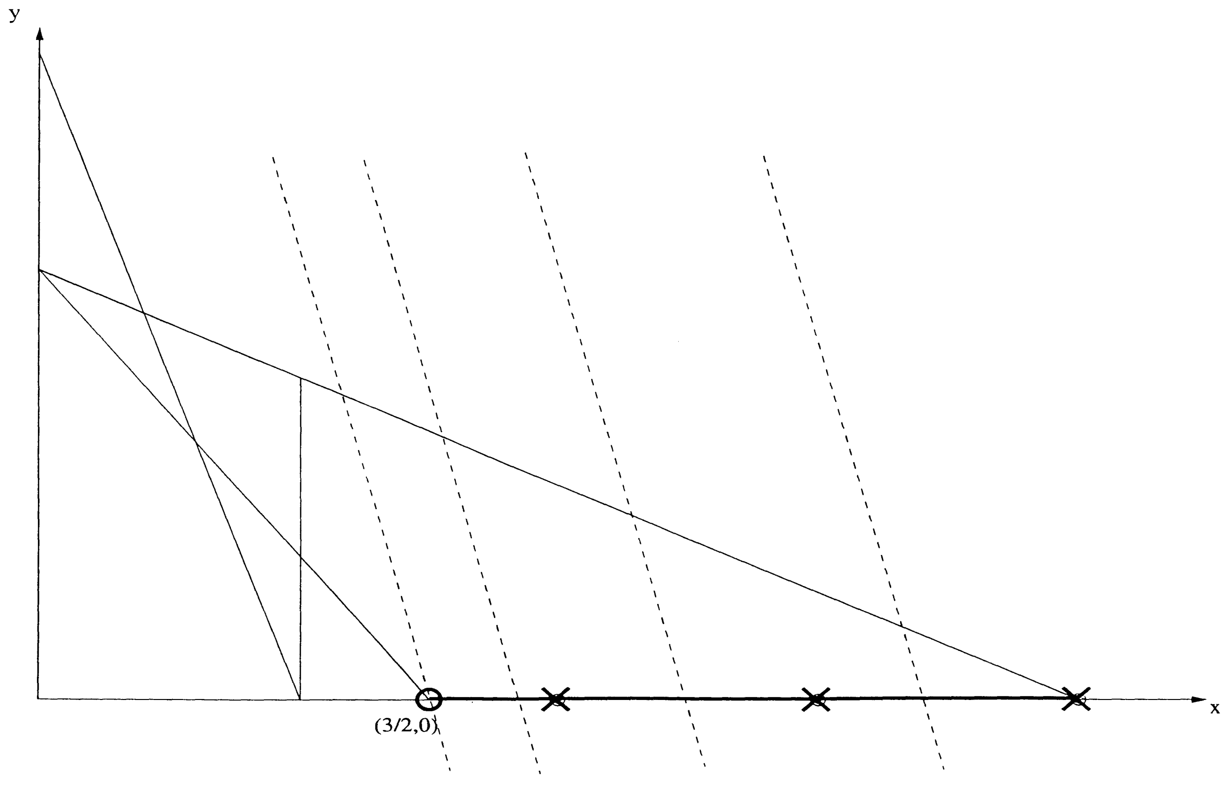
$$
\begin{aligned}
\text{Min} \quad & z = 4x + y \\
\text{subject to } & 3x + y \geq 3 \\
& 4x + 3y \geq 6 \\
& x + 2y \leq 4 \\
& x \geq 1 \\
& y \leq 0 \\
& x, y \geq 0, \text{ integer}
\end{aligned}
$$

and

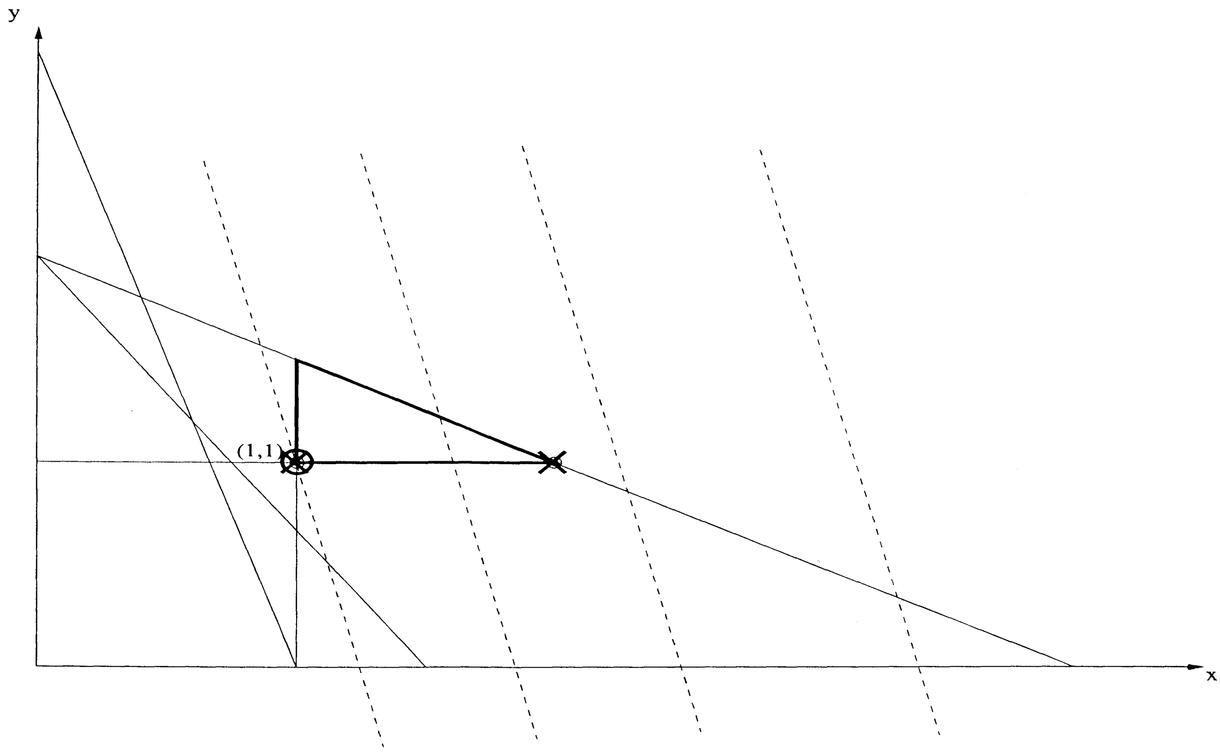
[P_4]

$$
\begin{aligned}
\text{Min} \quad & z = 4x + y \\
\text{subject to } & 3x + y \geq 3 \\
& 4x + 3y \geq 6 \\
& x + 2y \leq 4 \\
& x \geq 1 \\
& y \geq 1 \\
& x, y \geq 0, \text{ integer}
\end{aligned}
$$

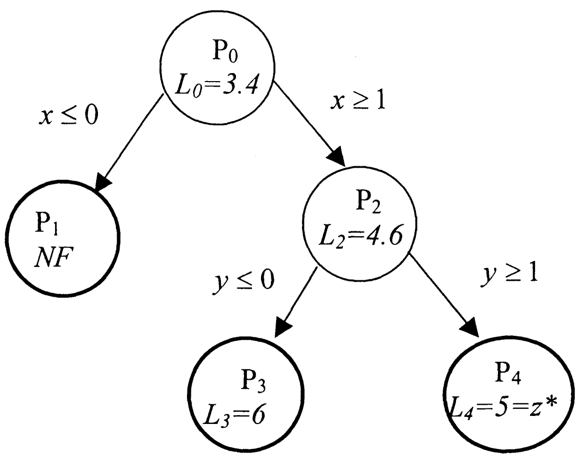
Extract now [P_3] and solve graphically its linear relaxation (next picture), finding the solution at (3/2,0) with value L_3 = 6.



Extract now [P4] and solve graphically its linear relaxation (next picture), finding the solution at (1,1) with value $L_4 = 5$.



The solution components are both integer, therefore the [P4] branch can be pruned. Moreover, this is the first time the incumbent can be update, that is $z^* = 5$. The latter allows also to prune the [P3] branch since, although no integer solution was found, still the lower bound L_3 of any other subproblem arising from [P3] is equal to 6 > 5. Therefore, since the problem list is now empty, and an incumbent solution was found, the B&B algorithm terminates yielding the optimal solution (1,1). The following tree describes the B&B iterations. A breadth first strategy has been used to explore the tree.



(ii) Rounding a non-integer solution to an integer one, based on some heuristics like rounding down, up or to the nearest integer point, may well give bad results in that the integer solution obtained might be either non-optimal or even infeasible.
In the previous problem we notice that the solutions obtained via integer rounding up, down and to the nearest are (1,2), (0,1) and (0,2). All these points can be seen lying outside the feasible region of the linear relaxation of the master program.

(b) First we define the variables of the problem.
Let $x_{ij} \geq 0$, for i=1,2,3 and j=1,2,3,4,5 denote the amount of goods to transport from the deposit i to the customer j. We also define binary variables y_i, for i=1,2,3, as follows:

$y_i = 1$, if the deposit i is built,
$y_i = 0$, otherwise.

Now we consider the objective function of the problem. It is requested to minimize the transportation cost, taking into account possible building costs of deposits, therefore the objective function is the following:

Minimize
$15x_{11} + 13x_{12} + 27x_{13} + 9x_{14} + 7x_{15} + 12x_{21} + 21x_{22} + 34x_{23} + 21x_{24} + 3x_{25} + 7x_{31} + 10x_{32} + 2x_{33} + 17x_{34} + 12x_{35} + 10{,}000y_1 + 15{,}000y_2 + 13{,}000y_3$

As for the constraints, we first focus on fulfillment of customer requests.

$$x_{11} + x_{21} + x_{31} = 91$$
$$x_{12} + x_{22} + x_{32} = 170$$
$$x_{13} + x_{23} + x_{33} = 135$$
$$x_{14} + x_{24} + x_{34} = 153$$
$$x_{15} + x_{25} + x_{35} = 110$$

Moreover, since at most two deposits may be constructed, we need to impose the following constraint:

$$y_1 + y_2 + y_3 \leq 2.$$

Finally, the links between x-variables and y-variables must be stated, taking into account the maximum capacity of each deposit:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 180\, y_1$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 230\, y_2$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 500\, y_3$$

The whole model is presented below.

Minimize

$$15x_{11} + 13x_{12} + 27x_{13} + 9x_{14} + 7x_{15} + 12x_{21} + 21x_{22} + 34x_{23} + 21x_{24} + 3x_{25} + 7x_{31} + 10x_{32} + 2x_{33} + 17x_{34} + 12x_{35} + 10{,}000y_1 + 15{,}000y_2 + 13{,}000y_3$$

Subject to:

$$x_{11} + x_{21} + x_{31} = 91$$
$$x_{12} + x_{22} + x_{32} = 170$$
$$x_{13} + x_{23} + x_{33} = 135$$
$$x_{14} + x_{24} + x_{34} = 153$$
$$x_{15} + x_{25} + x_{35} = 110$$

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 180\, y_1$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 230\, y_2$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 500\, y_3$$

$$y_1 + y_2 + y_3 \leq 2$$

$x_{ij} \geq 0$, i=1,2,3; j=1,2,3,4,5;
$y_i \in \{0,1\}$, i=1,2,3.

4.

(a) Min x_2

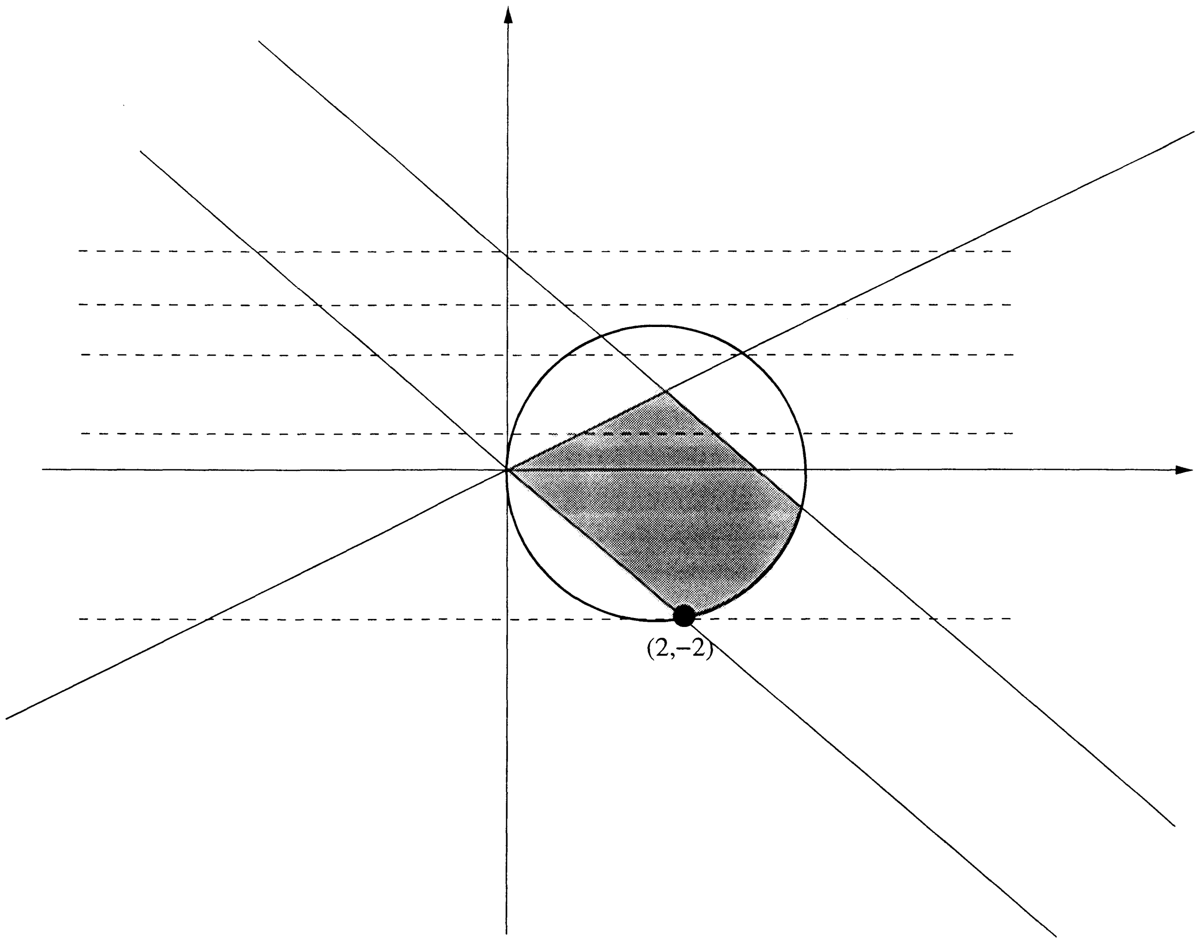
subject to $(x_1 - 2)^2 + x_2^2 \leq 4$

$x_1 - 2x_2 \geq 0$

$x_1 + x_2 \leq 3$

$x_1 + x_2 \geq 0$

(i) The solution can be found graphically at the point (2,–2).



(ii) The problem is convex. In fact, the objective function is linear, hence convex (and this makes unnecessary any consideration on whether it is a "min" or a "max" problem). As for the constraints, we notice that second, third and forth constraints are linear, hence convex. We focus then on the first constraint $(x_1 - 2)^2 + x_2^2 \leq 4$, that is $(x_1 - 2)^2 + x_2^2 - 4 = g(x_1, x_2) \leq 0$. Observe now that the Hessian matrix of g is

$$\nabla^2 g(x) = \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}$$

and therefore, since it is positive semidefinite for any x, then g is convex. Convexity of the first constraint can also be stated by recognizing that any pair (x_1,x_2) feasible for the first constraint must lie in a disc, which is a convex set.

(iii) We observe that the active constraints at the solution (2,–2) are

$$(x_1 - 2)^2 + x_2^2 - 4 \leq 0 \text{ and } -x_1 - x_2 \leq 0.$$

The active constraint gradients are (2,–4) and (–1, –1), respectively. Since they are linearly independent, then a regularity condition (namely LICQ) is satisfied at the solution.

(iv) Let $f(x_1,x_2) = x_2$, $g_1(x_1,x_2) = (x_1 - 2)^2 + x_2^2 - 4$, $g_2(x_1,x_2) = -x_1 + 2x_2$, $g_3(x_1,x_2) = x_1 + x_2 - 3$, and $g_4(x_1,x_2) = -x_1 - x_2$, then the problem can be written as follows

$$\begin{array}{ll} \text{Min} & f(x_1,x_2) \\ \text{subject to} & g_1(x_1,x_2) \leq 0 \\ & g_2(x_1,x_2) \leq 0 \\ & g_3(x_1,x_2) \leq 0 \\ & g_4(x_1,x_2) \leq 0. \end{array}$$

The KKT conditions for this problem are stated below:

$$\begin{array}{l} \nabla f(x_1,x_2) + \nabla g_1(x_1,x_2)\,\lambda_1 + \nabla g_2(x_1,x_2)\,\lambda_2 + \nabla g_3(x_1,x_2)\,\lambda_3 + \nabla g_4(x_1,x_2)\,\lambda_4 = 0 \\ g_1(x_1,x_2)\,\lambda_1 + g_2(x_1,x_2)\,\lambda_2 + g_3(x_1,x_2)\,\lambda_3 + g_4(x_1,x_2)\,\lambda_4 = 0 \\ g_1(x_1,x_2) \leq 0 \\ g_2(x_1,x_2) \leq 0 \\ g_3(x_1,x_2) \leq 0 \\ g_4(x_1,x_2) \leq 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0. \end{array}$$

Since the complementarity condition implies that $g_i(x_1,x_2)\,\lambda_i = 0$ for every i=1,..,4, then inactivity of the second constraint as well as of the third one at the solution (2,–2), imply that both $\lambda_2 = 0$ and $\lambda_3 = 0$. Therefore, taking into account that (2,–2) is a feasible point, namely that $g_i(x_1,x_2) \leq 0$ for every i=1,..,4, and plugging (λ_2, λ_3)=(0,0) into the system above, it remains to check feasibility of the following system

$$\begin{array}{c} \nabla f(x_1,x_2) + \nabla g_1(x_1,x_2)\,\lambda_1 + \nabla g_4(x_1,x_2)\,\lambda_4 = 0 \\ \lambda_1, \lambda_4 \geq 0. \end{array}$$

Notice that the first row amounts solving the following system of linear equalities

$$\begin{array}{c} 2\,\lambda_1 - \lambda_4 = 0 \\ -4\,\lambda_1 - \lambda_4 = -1 \end{array}$$

which admits the unique solution (λ_1, λ_4)=(1/6,2/6), and since the two components are positive they fulfill the remaining sign constraints as well. Hence, the solution (2,–2) satisfies the KKT conditions.

(b) Min $c^T x$
subject to $Ax = b$
$x \geq 0$

The KKT conditions are the following:

$c - A^T\lambda - s = 0;$
$Ax - b = 0;$
$x \geq 0;$
$s \geq 0;$
$s^T x = 0.$

Consider now any feasible solution of the linear program, say $x^{\#}$. The following, taking into account the KKT system and feasibility of $x^{\#}$, holds:

$$c^T x^{\#} = (A^T\lambda^* + s^*)^T x^{\#} = (\lambda^*)^T b + (s^*)^T x^{\#} \geq (\lambda^*)^T b.$$

Notice that $(\lambda^*)^T b = (\lambda^*)^T Ax^* = ((\lambda^*)^T A + s^*)^T x^* = c^T x^*$. Therefore for any feasible $x^{\#}$ for LP it holds that $c^T x^{\#} \geq c^T x^*$, that is x^* is a global optimum for LP.

END OF SOLUTION