PART IIA 2005
3F1: Signals and systems
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Datasheet: None

Module 3F1, April 2005 – SIGNALS AND SYSTEMS – Solutions

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(a) From the pole-zero diagram in Figure 1 we see that all the poles are inside the unit disk, and so the system is stable.

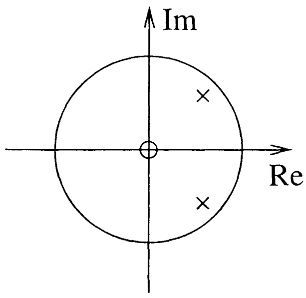

Figure 1: Pole-zero diagram

[20%]

(b) The magnitude response is given by $|H(e^{j\theta})|$ as θ varies from 0 to π. Thus,

$$|H(e^{j\theta})| = \frac{1}{|e^{j\theta}-p|\,|e^{j\theta}-\bar{p}|} = \frac{1}{d_1 d_2} \quad \text{where } d_1 = \left|e^{j\theta}-p\right| \text{ and } d_2 = \left|e^{j\theta}-\bar{p}\right|$$

See Figure 2. As θ varies from 0 to π, we see that d_1 gets smaller and smaller as it approaches $\theta = \pi/4$. Note that compared with d_1, d_2 does not change that much in that interval. d_1 is smallest at $\theta = \pi/4$ and therefore, the maximum magnitude response happens at approximately $\theta = \pi/4$ (the approximately comes from neglecting the small changes of d_2 around $\theta = \pi/4$).

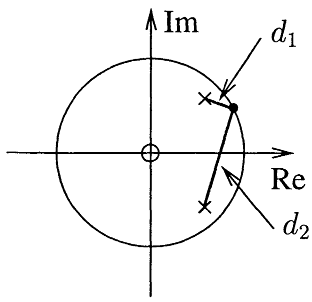

Figure 2: Pole-zero diagram

[20%]

(c) Assuming $\theta_{max} \approx \frac{\pi}{4}$:

$$A_{max} \approx \frac{1}{|e^{j\pi/4}-p|\,|e^{j\pi/4}-\bar{p}|} \approx \frac{1}{0.1\sqrt{0.9^2+1^2}} = \frac{10}{\sqrt{1.81}}$$

For the phase angle, we sum the contributions of the zeros and subtract the contributions of the poles. Thus,

$$\Phi_{max} \approx \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

[30%]

(d) $y_k = 2A_{max}\cos(\theta_{max}k + \Phi_{max}) \approx \frac{20}{\sqrt{1.81}}\cos(\frac{\pi}{4}k - \frac{\pi}{4})$. [10%]

(e) As mention in part (b), the contribution of the pole $\bar{p}$ near the frequency $\theta = \pi/4$ is small when compared with the contribution of the pole p. Another approximation is that the unit circle can be approximated by a line for small changes around $\theta = \pi/4$ (see Figure 3). The magnitude near $\pi/4$ is given by

$$|H(e^{j\theta})| = \frac{1}{|e^{j\theta} - p|\,|e^{j\theta} - \bar{p}|} \approx \frac{1}{|e^{j\theta} - p|}c_1$$

where c_1 is a constant. Therefore, we need to find θ such that

$$A_{max}\frac{1}{\sqrt{2}} = \frac{1}{|e^{j\theta} - p|}c_1$$

Since

$$A_{max} \approx \frac{1}{|e^{j\pi/4} - p|}c_1 = \frac{1}{0.1}c_1$$

we need to find θ such that

$$|e^{j\theta} - p| = 0.1\sqrt{2}$$

Approximating the unit circle by a line for small changes around θ (see Figure 3) we see that the bandwidth approximately covers the range $\pi/4 \pm 0.1$.

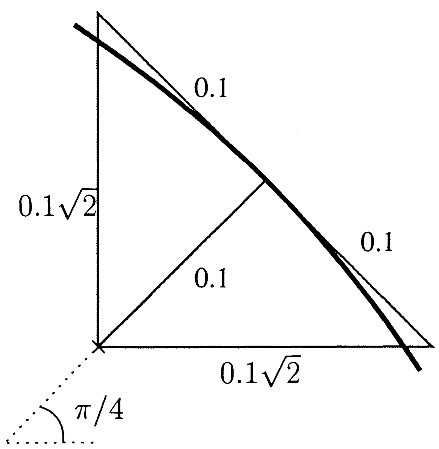

Figure 3: Pole-zero diagram

[20%]

2 (a) **Discrete-time Control System**

(i) The closed-loop transfer function is given by

$$\frac{Y(z)}{U(z)} = \frac{\frac{zK_1}{2z-1}}{1+\frac{zK_1}{2z-1}} = \frac{zK_1}{z(2+K_1)-1}$$

The system is stable if the closed-pole at $z = \frac{1}{2+K_1}$ lies inside the unit disc, i.e. if $K_1 < -3$ or $K_1 > -1$. [15%]

(ii) Since $E(z) = U(z) - Y(z)$

$$E(z) = \frac{2z-1}{z(2+K_1)-1}U(z)$$

In the z-domain the unit step is given by,

$$U(z) = \frac{z}{z-1}$$

which means that

$$E(z) = \frac{z}{z-1}\frac{2z-1}{z(2+K_1)-1}$$

Thus, the Final Value Theorem (FVT) applies and

$$\lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}\frac{z(2z-1)}{z(2+K_1)-1} = \frac{1}{K_1+1}$$

[25%]

(iii) For the steady-state error to be less than 1%,

$$|\lim_{k\to\infty} e_k| < 0.01$$

Therefore

$$|K_1+1| > 100$$

and so

$$K_1 > 99 \quad \text{or} \quad K_1 < -101$$

These values all lie within the range of values in (i), permitted for stability. [10%]

2 (b) **WSS Random Processes**

(i) $X(t)$ is defined to be *Wide Sense Stationary* (WSS) iff:

- The mean value is independent of t such that

$$E[X(t)] = \mu \quad \text{for all } t$$

- And the autocorrelation function depends only upon $\tau = t_2 - t_1$ such that

$$r_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)] = E[X(t_1)\, X(t_1 + \tau)] = r_{XX}(\tau) \quad \text{for all } t_1$$

WSS is used when we are only interested in the properties of moments up to 2nd order (mean, autocorrelation, covariance etc.). [10%]

(ii) For the given function, U and V have zero mean, so X will also have zero mean for all t. Hence the first condition for WSS is satisfied.

The second condition requires calculation of the ACF, which is given by the following expectations over α:

$$\begin{aligned}
r_{XX}(t_1, t_2) &= E[X(t_1, \alpha)\, X(t_2, \alpha)] \\
&= E[\{U\cos(\omega_0 t_1) + V\sin(\omega_0 t_1)\}\ \{U\cos(\omega_0 t_2) + V\sin(\omega_0 t_2)\}] \\
&= E[U^2]\cos(\omega_0 t_1)\cos(\omega_0 t_2) + E[V^2]\sin(\omega_0 t_1)\sin(\omega_0 t_2) \\
&\quad + E[UV]\{\cos(\omega_0 t_1)\sin(\omega_0 t_2) + \sin(\omega_0 t_1)\cos(\omega_0 t_2)\} \\
&= \sigma_U^2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sigma_V^2\sin(\omega_0 t_1)\sin(\omega_0 t_2) + 0
\end{aligned}$$

The first two terms are valid because U and V have zero means, and the final term is zero because U and V also are independent. Since by assumption $\sigma_U = \sigma_V$ and converting the products of sines and cosines into sums gives:

$$\begin{aligned}
r_{XX}(t_1, t_2) &= \sigma_U^2\{\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sin(\omega_0 t_1)\sin(\omega_0 t_2)\} \\
&= \sigma_U^2\cos(\omega_0 t_1 - \omega_0 t_2)
\end{aligned}$$

which means that X is WSS since r_{XX} depends only on $\tau = t_2 - t_1$. Thus, the ACF simplifies to: $r_{XX}(\tau) = \sigma_U^2\cos(\omega_0 \tau)$ [40%]

3 (a) **Ergodic Processes**

In an Ergodic Random Process we can exchange *Ensemble Averages* for *Time Averages*. This is equivalent to assuming that our ensemble of random signals is just composed of all possible time shifts of a single signal $X(t)$. [10%]

(b) **ACF of system output**

The linear system with input $X(t)$ and output $Y(t)$ has an impulse response $h(t)$, so

$$Y(t) = h(t) * X(t) = \int h(\beta)\, X(t-\beta)\, d\beta$$

[Note: all integrals are assumed to have limits from $-\infty$ to $+\infty$, unless shown otherwise.]

Then the ACF of Y is

$$\begin{aligned} r_{YY}(t_1, t_2) &= E\left[Y(t_1)\, Y(t_2)\right] \\ &= E\left[\left(\int h(\beta_1) X(t_1-\beta_1)\, d\beta_1\right)\left(\int h(\beta_2) X(t_2-\beta_2)\, d\beta_2\right)\right] \\ &= E\left[\int\int h(\beta_1)\, h(\beta_2)\, X(t_1-\beta_1)\, X(t_2-\beta_2)\, d\beta_1\, d\beta_2\right] \\ &= \int\int h(\beta_1)\, h(\beta_2)\, E\left[X(t_1-\beta_1)\, X(t_2-\beta_2)\right]\, d\beta_1\, d\beta_2 \\ &= \int\int h(\beta_1)\, h(\beta_2)\, r_{XX}(t_1-\beta_1, t_2-\beta_2)\, d\beta_1\, d\beta_2 \end{aligned}$$

If X is WSS, then we substitute $\tau = t_2 - t_1$ and $t = t_1$ to get

$$\begin{aligned} r_{YY}(\tau) &= E\left[Y(t)\, Y(t+\tau)\right] \\ &= \int\int h(\beta_1)\, h(\beta_2)\, r_{XX}(\tau + \beta_1 - \beta_2)\, d\beta_1\, d\beta_2 \end{aligned}$$

Now we can substitute for $r_{XX}(\tau) = \rho\, \delta(\tau)$ and use the sifting property of the δ function to get

$$\begin{aligned} r_{YY}(\tau) &= \int\int h(\beta_1)\, h(\beta_2)\, \rho\, \delta(\tau + \beta_1 - \beta_2)\, d\beta_2\, d\beta_1 \\ &= \rho \int h(\beta_1)\, h(\tau + \beta_1)\, d\beta_1 \end{aligned}$$

[Note: This integral represents $h(\tau)$ convolved with $h(-\tau)$.] [30%]

(c) **ACF for a system with an exponential response**

If $h(t) = \begin{cases} \frac{1}{T}\exp(-t/T) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

then we must deal with the discontinuity in h by suitable treatment of the limits of the above integral. Both terms of the product in the integral must be non-zero to make any contribution to the result.

First consider the case when $\tau \geq 0$. In this case, both terms are non-zero when $\beta_1 \geq 0$. Therefore

$$\begin{aligned}
r_{YY}(\tau) &= \rho \int h(\beta_1)\, h(\tau + \beta_1)\, d\beta_1 \\
&= \frac{\rho}{T^2} \int_0^\infty \exp\left(\frac{-\beta_1}{T}\right) \exp\left(\frac{-\tau - \beta_1}{T}\right) d\beta_1 \\
&= \frac{\rho}{T^2} \int_0^\infty \exp\left(\frac{-\tau - 2\beta_1}{T}\right) d\beta_1 \\
&= \frac{-\rho}{2T} \left[\exp\left(\frac{-\tau - 2\beta_1}{T}\right)\right]_0^\infty = \frac{\rho}{2T} \exp\left(\frac{-\tau}{T}\right) \qquad \text{if } \tau \geq 0
\end{aligned}$$

Now consider $\tau < 0$. In this case, both terms in the integral are non-zero only when $\beta_1 \geq -\tau$. Therefore

$$\begin{aligned}
r_{YY}(\tau) &= \frac{\rho}{T^2} \int_{-\tau}^\infty \exp\left(\frac{-\beta_1}{T}\right) \exp\left(\frac{-\tau - \beta_1}{T}\right) d\beta_1 \\
&= \frac{-\rho}{2T} \left[\exp\left(\frac{-\tau - 2\beta_1}{T}\right)\right]_{-\tau}^\infty = \frac{\rho}{2T} \exp\left(\frac{\tau}{T}\right) \qquad \text{if } \tau < 0
\end{aligned}$$

Combining these two results, we get

$$r_{YY}(\tau) = \frac{\rho}{2T} \exp\left(\frac{-|\tau|}{T}\right)$$

so that $r_{YY}(\tau)$ is symmetrical about $\tau = 0$, as expected for an ACF. [30%]

(d) **Power Spectral Density of** Y

The *Power Spectral Density* (PSD) of a random process is defined to be the Fourier Transform of its ACF. Therefore the PSD of Y is given by

$$\begin{aligned}
\mathcal{S}_Y(\omega) &= \text{FT}\{r_{YY}(\tau)\} = \int_{-\infty}^\infty r_{YY}(\tau) \exp(-j\omega\tau)\, d\tau \\
&= \int_{-\infty}^\infty \frac{\rho}{2T} \exp\left(\frac{-|\tau|}{T}\right) \exp(-j\omega\tau)\, d\tau \\
&= \int_{-\infty}^0 \frac{\rho}{2T} \exp\left(\frac{\tau}{T}\right) \exp(-j\omega\tau)\, d\tau + \int_0^\infty \frac{\rho}{2T} \exp\left(\frac{-\tau}{T}\right) \exp(-j\omega\tau)\, d\tau \\
&= \int_0^\infty \frac{\rho}{2T} \exp\left(\frac{-\tau}{T}\right) \exp(j\omega\tau)\, d\tau + \int_0^\infty \frac{\rho}{2T} \exp\left(\frac{-\tau}{T}\right) \exp(-j\omega\tau)\, d\tau \\
&= \frac{\rho}{2T} \int_0^\infty \exp\left(\frac{-\tau(1 - j\omega T)}{T}\right) + \exp\left(\frac{-\tau(1 + j\omega T)}{T}\right) d\tau \\
&= \frac{\rho}{2T} \left[\frac{-T}{1 - j\omega T} \exp\left(\frac{-\tau(1 - j\omega T)}{T}\right) + \frac{-T}{1 + j\omega T} \exp\left(\frac{-\tau(1 + j\omega T)}{T}\right)\right]_0^\infty \\
&= \frac{\rho}{2T} \left[\frac{T}{1 - j\omega T} + \frac{T}{1 + j\omega T}\right] = \frac{\rho}{2} \frac{1 + j\omega T + 1 - j\omega T}{1 + \omega^2 T^2} = \frac{\rho}{1 + \omega^2 T^2}
\end{aligned}$$

[30%]

4 (a) The mutual information can be defined mathematically in a number of equivalent ways:

$$I(A;B) = H(A) - H(A|B) = H(B) - H(B|A) = H(A) + H(B) - H(A,B)$$

where $H(A|B)$ is the expected value of the entropy of A given that the value of B is known. In english, the mutual information is the amount of information that knowing one of the variables gives about the other. [15%]

(b)

$$H(S_i) = -\left(0.7\log_2(0.7) + 0.2\log_2(0.2) + 0.07\log_2(0.07) + 0.03\log_2(0.03)\right) = 1.245$$

To calculate the entropy of X_i the probabilities that $X_i = 0$ and 1 are needed:

$$P(X_i = 0) = 0.7 * 1 + 0.2 * 0.5 = 0.8$$

$$P(X_i = 1) = 0.2$$

Hence

$$H(X_i) = -\left(0.8\log_2(0.8) + 0.2\log_2(0.2)\right) = 0.722$$

$H(X_i|S)_i$ is simple to calculate since it is only non zero for $S_i = B$.

$$H(X_i|S_i) = 0 * 0.7 + 1 * 0.2 + 0 * 0.07 + 0 * 0.03 = 0.2$$

$$I(X_i; S_i) = 0.722 - 0.2 = 0.522$$

[35%]

(c)

```
A 0.7 ----------------------- 1.0
B 0.2 -------------- 0.3 --/
C 0.07 ---- 0.1 --/
D 0.03 --/
```

giving the code

```
A 0
B 10
C 110
D 111
```

To calculate the efficiency of the code, the average code word length is needed:

$$L = 0.7 * 1 + 0.2 * 2 + 0.07 * 3 + 0.03 * 3 = 1.4$$

The efficiency is then given by

$$\eta = 1.245/1.4 = 0.8893$$

If the source were extended to order four, the efficiency of the code should increase and tend towards unity, since it cannot decrease. The source S contains a probability greater than 0.5, which will encoded rather inefficiently by an order-one Huffman code. The most probable event for an order-four code is AAAA which has probability $0.7^4 = 0.24$ and is likely to be encoded with very high efficiency using 2 bits for the 4 symbols. [30%]

(d) If the statistics in the tables do not change but the even X always equals the odd X, then this implies that S and X both have memory.

In particular, $H(X_2|X_1) = 0$.

Hence we know how X_2 relates to X_1, but we do not directly know how S_2 relates to S_1. So we have to calculate the joint entropy, by reference back to X_1, using the mutual information calculated in part (b).

Hence

$$\begin{aligned}
H(S_1, S_2, X_1, X_2) &= H(S_1, S_2, X_1) \quad \text{since } X_2 = X_1 \\
&= H(X_1) + H(S_1|X_1) + H(X_2|X_1) + H(S_2|X_2) \\
&= H(X_1) + (H(S_1) - I(S_1; X_1)) + 0 + (H(S_2) - I(S_2; X_2)) \\
&= 0.722 + (1.245 - 0.522) + (1.245 - 0.522) \\
&= 2.168
\end{aligned}$$

[20%]