A.1 (a) $e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$

(b) From (a), $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$

$$\therefore \frac{d}{dt}\{e^{At}\} = A + \frac{2}{2!}A^2 t + \frac{3}{3!}A^3t^2 + \cdots$$

$$= A\left(I + At + \frac{1}{2!}A^2t^2 + \cdots\right)$$

$$= Ae^{At}.$$

If $x(t) = e^{At}x(0)$ then

$$\dot{x}(t) = Ae^{At}x(0) = Ax(t),$$

which verifies that $x(t) = e^{At}x(0)$ is a solution of $\dot{x} = Ax$.

(c) Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $\dot{x}_1 = x_2$, $\dot{x}_2 = u$

is the same as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

so $\underline{\underline{A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}}$ and $\underline{\underline{B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}}$

$$\therefore A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and clearly $A^k = 0$ for $k \geq 2$.

$$\therefore e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots = I + At$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}}$$

A.1 cont'd.

(d) The general solution to $\dot{x} = Ax + Bu$

is $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$

Taking initial conditions at $t = kT$ instead of 0 gives

$$x_{k+1} = e^{AT} x_k + \int_{kT}^{(k+1)T} e^{A([k+1]T-\tau)} Bu(\tau)d\tau$$

$$= e^{AT} x_k + e^{A(k+1)T} \int_{kT}^{(k+1)T} e^{-A\tau} d\tau \, Bu_k$$

since $u(\tau) = u_k$ for $kT \leq \tau < (k+1)T$.

Now $\int_{kT}^{(k+1)T} e^{-A\tau} d\tau = \int_{kT}^{(k+1)T} \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} d\tau$ (from (c))

$$= \left[\begin{pmatrix} \tau & -\tau^2/2 \\ 0 & \tau \end{pmatrix} \right]_{kT}^{(k+1)T}$$

$$= \begin{pmatrix} T & \frac{-(k+1)^2T^2 + k^2T^2}{2} \\ 0 & T \end{pmatrix}$$

$$= \begin{pmatrix} T & (-2k+1)\frac{T^2}{2} \\ 0 & T \end{pmatrix}$$

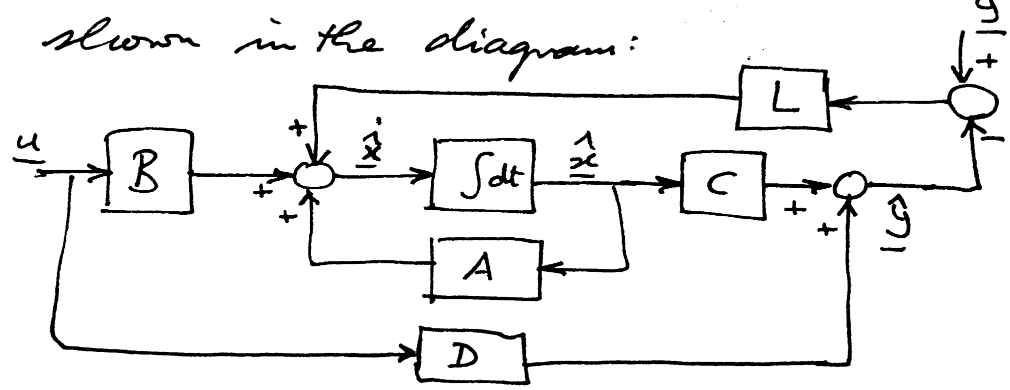
$$\therefore \underline{\underline{F = e^{AT} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}}} \quad \text{and}$$

$$G = e^{A(k+1)T} \begin{pmatrix} T & (-2k+1)\frac{T^2}{2} \\ 0 & T \end{pmatrix} B = \begin{pmatrix} 1 & (k+1)T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T & (-2k+1)\frac{T^2}{2} \\ 0 & T \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & (k+1)T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (-2k+1)\frac{T^2}{2} \\ T \end{pmatrix} = \underline{\underline{\begin{pmatrix} T^2/2 \\ T \end{pmatrix}}}$$

A.2

(a) Observer has the structure shown in the diagram:



and its equation of motion is

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x} - Du)$$

It is used to estimate $\hat{x}$ if x cannot be measured directly.

(b) We have the equation for $\dot{\hat{x}}$ above, and $\dot{x} = Ax + Bu$

Let $e = x - \hat{x}$.

Then $\dot{e} = \dot{x} - \dot{\hat{x}}$

$$= [Ax + Bu] - [A\hat{x} + Bu + L(y - C\hat{x} - Du)]$$

$$= Ae - Ly + LC\hat{x} + LDu$$

But $y = Cx + Du$

hence $\dot{e} = Ae - L(Cx + Du) + LC\hat{x} + LDu$

$$= Ae - LCe$$

$$= (A - LC)e$$

Hence $e(t) = \exp([A - LC]t)e(0)$, so that $e(t) \to 0$ if all the eigenvalues of $A - LC$ have negative real parts.

A 2 cont'd. (c) If $\underline{x} = \begin{bmatrix} \delta \\ b \\ \psi_c \end{bmatrix}$ we have

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\delta} \\ \dot{b} \\ \dot{\psi}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/T \end{bmatrix}\begin{bmatrix} \delta \\ b \\ \psi_c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/T \end{bmatrix}\psi$$

since $\dot{\delta} = \dot{\psi}_G - \dot{\psi} = b$, $\dot{b} = 0$

and $\dot{\psi}_c = -\frac{1}{T}\psi_c + \frac{1}{T}\psi$

Hence A is verified and $B = \begin{bmatrix} 0 \\ 0 \\ 1/T \end{bmatrix}$.

Also $\underline{y} = \begin{bmatrix} \psi_G \\ \psi_c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \delta \\ b \\ \psi_c \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\psi$

since $\psi_G = \delta + \psi$.

Hence C is verified and $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

If $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & 0 \\ 0 & l_{32} \end{bmatrix}$

then $LC = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & 0 & 0 \\ 0 & 0 & l_{32} \end{bmatrix}$

A. 2 cont'd and hence

$$A - LC = \begin{bmatrix} -l_{11} & 1 & 0 \\ -l_{21} & 0 & 0 \\ 0 & 0 & -\frac{1}{T} - l_{32} \end{bmatrix}$$

Therefore

$$\det(\lambda I - [A - LC]) = \det \begin{bmatrix} \lambda + l_{11} & -1 & 0 \\ l_{21} & \lambda & 0 \\ 0 & 0 & \lambda + \frac{1}{T} + l_{32} \end{bmatrix}$$

$$= \left(\lambda + \frac{1}{T} + l_{32}\right) \det \begin{bmatrix} \lambda + l_{11} & -1 \\ l_{21} & \lambda \end{bmatrix}$$

$$= \left(\lambda + \frac{1}{T} + l_{32}\right) \left[\lambda(\lambda + l_{11}) + l_{21} \right]$$

$$= \left(\lambda + \frac{1}{T} + l_{32}\right)\left(\lambda^2 + l_{11}\lambda + l_{21}\right)$$

So l_{11}, l_{21}, l_{32} can be chosen to give any (monic, real) polynomial of degree 3, and hence can be chosen to give any desired set of eigenvalues.

Benefits of observer for this system:

An accurate estimate of Ψ can be obtained (ie corrected for b and δ), in principle arbitrarily quickly (since the observer poles can be chosen arbitrarily). Thus the lag of the compass can be overcome. Also the estimate of b allows a corrected estimate of the heading rate $\dot{\Psi}$ to be obtained, should that be needed.

A. 3 (a) $\dot{\delta} = u, \quad \dot{\phi} = r, \quad \dot{\psi} = \frac{v}{\rho}\delta,$

$$\ddot{\phi} = \dot{r} = \frac{g}{h}\phi + \frac{v^2}{ph}\delta + \frac{bv}{ph}\, u$$

So

$$\dot{\underline{x}} = \begin{bmatrix} \dot{\delta} \\ \dot{\psi} \\ \dot{\phi} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ v/p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ v^2/ph & 0 & g/h & 0 \end{bmatrix} \begin{bmatrix} \delta \\ \psi \\ \phi \\ r \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ bv/ph \end{bmatrix} u$$

which is in the form

$$\dot{\underline{x}} = A\,\underline{x} + \underline{b}\,\omega.$$

(b) Stability is determined by the eigenvalues of 'A':

<u>Either</u> Note that $A = \begin{bmatrix} W & 0 \\ X & Y \end{bmatrix}$ and hence its eigenvalues are the union of those of W and Y.

$$W: \begin{vmatrix} \lambda & 0 \\ -v/p & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow$$

eigenvalues of W are 0, 0.

$$Y: \begin{vmatrix} \lambda & -1 \\ -g/h & \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \frac{g}{h} = 0$$

$$\Rightarrow \left(\lambda - \sqrt{g/h}\right)\left(\lambda + \sqrt{g/h}\right) = 0$$

$\Rightarrow$ eigenvalues of Y are $\pm\sqrt{g/h}$.

$\therefore$ eigenvalues of A are $\underline{0,\ 0,\ +\sqrt{g/h},\ -\sqrt{g/h}}$.

<u>Or</u> $\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 0 & 0 \\ -v/p & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ -v^2/ph & 0 & -\frac{g}{h} & \lambda \end{bmatrix}$

A. 3 cont'd.

$$\therefore \det(\lambda I - A) = \lambda \det \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -g/h & \lambda \end{bmatrix}$$

$$= \lambda^2 \begin{vmatrix} \lambda & -1 \\ -g/h & \lambda \end{vmatrix} = \lambda^2(\lambda^2 - g/h) = 0$$

if $\underline{\lambda = 0 \text{ or } \lambda = 0 \text{ or } \lambda = \pm\sqrt{g/h}}$.

Hence the bicycle is open-loop <u>unstable</u>.

(c) Controllability: Let $P = [\underline{b}, A\underline{b}, A^2\underline{b}, A^3\underline{b}]$.

Then
$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v/p & 0 & 0 \\ 0 & \frac{bv}{ph} & \frac{v^2}{ph} & \frac{gbv}{ph^2} \\ \frac{bv}{ph} & \frac{v^2}{ph} & \frac{gbv}{ph^2} & \frac{gv^2}{ph^2} \end{bmatrix}$$

NB: Evaluate $A^2\underline{b}$ as $A(A\underline{b})$ not $(AA)(\underline{b})$. Much quicker!

$$\det P = 1 \times \frac{v}{p} \times \begin{vmatrix} \frac{v^2}{ph} & \frac{gbv}{ph^2} \\ \frac{gbv}{ph^2} & \frac{gv^2}{ph^2} \end{vmatrix}$$

$$= \frac{v}{p} \times \left(\frac{v}{ph}\right)^2 \begin{vmatrix} v & \frac{gb}{h} \\ \frac{gb}{h} & \frac{gv}{h} \end{vmatrix}$$ (taking $\frac{v}{ph}$ out as a factor)

$$= \frac{v}{p} \times \left(\frac{v}{ph}\right)^2 \times \left(\frac{gv^2}{h} - \frac{g^2b^2}{h^2}\right)$$

$$= \frac{v^3}{p^3h^2} \times \frac{g}{h}\left(v^2 - \frac{gb^2}{h}\right)$$

$\neq 0$ if $\underline{v \neq 0}$ and $v^2 \neq \frac{gb^2}{h}$

$\Downarrow$

$\text{rank}(P) = 4$

$\Downarrow$

<u>Controllable</u>

i.e. $\underline{v \neq b\sqrt{g/h}}$

A. 3 (d) Since the bicycle is open-loop unstable, successful control is only possible if the unstable modes are observable from the sensed outputs. So examine the observability from ψ and $\dot{\psi}$. This corresponds to the following 'C' matrix:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{Let } Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

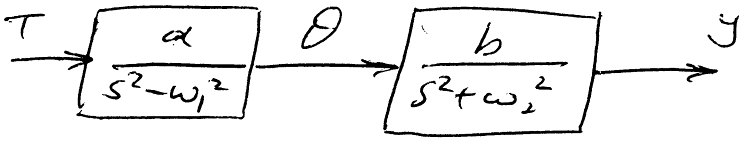
$$\text{Then } Q = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline v/p & 0 & 0 & 0 \\ v^2/ph & 0 & g/h & 0 \\ \hline \vdots & \vdots & \vdots & \vdots \end{array}\right] \begin{array}{l} \}\, C \\ \\ \}\, CA \\ \\ \end{array}$$

Clearly the first 4 rows of Q are linearly independent (if $v \neq 0$!) so rank$(Q) = 4$ hence the state is <u>observable</u>.

Therefore it is possible to stabilise the bicycle by sensing ψ and r, and (from (c)) by using u as the control input.

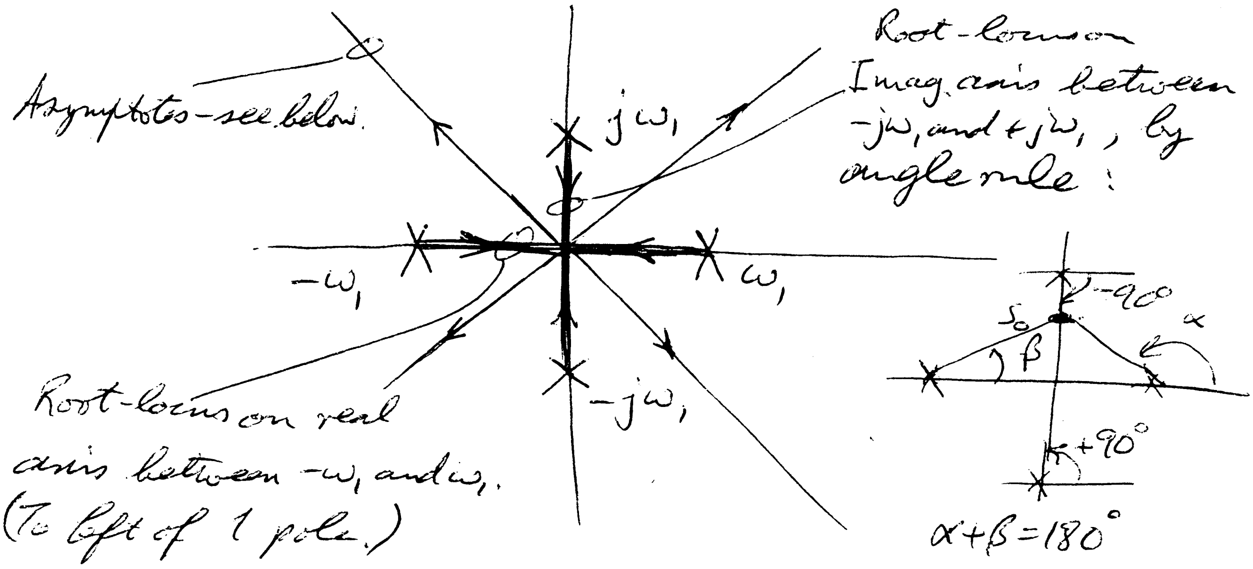
There is the further question of whether the steady-state can be held with the available inputs. Since the 'set-point' here is $(\psi \equiv 0, \phi \equiv 0)$ this can be achieved by $(\delta = 0, u = 0)$ — as can be seen from the equations.

A.4 (a)



$$L(s) = \frac{ab}{(s^2-\omega_1^2)(s^2+\omega_2^2)}$$

Root-locus diagram (if $\omega_1 = \omega_2$):



poles − # zeros = 4 − 0 = 4

∴ 4 asymptotes, making angles $\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$ for $n = 0, 1, 2, 3$, with real axis.

Asymptotes intersect real axis at 'CG', ie

$$\frac{-\omega_1 + \omega_1 - j\omega_1 + j\omega_1}{4} = 0 \quad \text{ie the origin.}$$

Using the hint, the root-locus can now be completed – the R.L. lies on the asymptotes, and on the axes as stated above.

[Not using the hint: $(s^2-\omega_1^2)(s^2+\omega_1^2) + K$

$= s^4 - \omega_1^4 + K$

So $K = \omega_1^4$ gives $s^4 = 0$ ie all 4 roots at 0,
so the root-locus does not branch on the axes except at 0.]

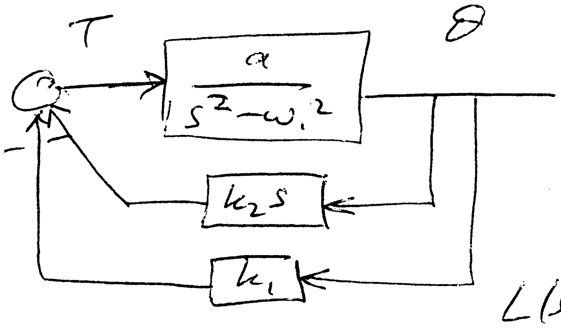
A.4 cont'd.

Proportional feedback won't work:

'Hence' proof: The root-locus shows that at least one closed-loop pole is always in the right-half plane.

'Otherwise' proof: Closed-loop characteristic polynomial is $s^4 - \omega_1^4 + K = 0$.
Hence Routh-Hurwitz criterion is violated.
(Or can find roots explicitly for cases $K - \omega_1^4 > 0$ and $K - \omega_1^4 < 0$.)

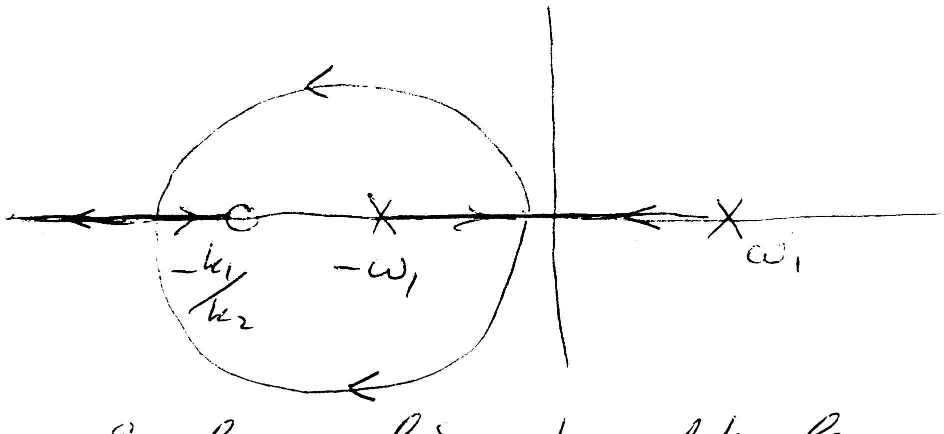
(b)



This feedback loop has return-ratio

$$L(s) = \frac{a(k_1 + k_2 s)}{s^2 - \omega_1^2}$$

hence has a zero at $-k_1/k_2$. If $-k_1/k_2 < -\omega_1$ then the root-locus diagram is



So by making k_1 and k_2 large enough, and $-k_1/k_2 < -\omega_1$, closed-loop stability can be achieved.

Alternative argument: Closed-loop characteristic polynomial is $(s^2 - \omega_1^2) + a(k_1 + k_2 s) = s^2 + ak_2 s + (ak_1 - \omega_1^2)$ which has two roots in LHP if $ak_2 > 0$ and $ak_1 - \omega_1^2 > 0$.

A.4 contd.

(c) From the root-locus diagram it is clear that closed-loop stability is obtained for $0 < k < k_1$, where k_1 is the gain at which the root-locus crosses the imaginary axis.

From the figure, the crossing points can be estimated to be $\pm j5.5$ (Exact value: 5.48).

So

$$1 + k_1 L(j5.5) = 0$$

or

$$k_1 = \frac{1}{|L(j5.5)|}$$

$$= \frac{|j5.5+6|^2 \times |j5.5+2j| \times |j5.5-2j|}{|j5.5+0.5|}$$

$$= \frac{66.25 \times 7.5 \times 3.5}{5.52}$$

$$= \underline{315.05} \qquad \text{(Exact value: 312)}$$

$\therefore$ closed-loop is stable for $\underline{\underline{0 < k < 315.05}}$

Any value reasonably close to 312 is acceptable here, if supported by arguments!