PART IIA 2005
3F3: Signal and pattern processing
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***Datasheet:* None**

3F3 Questions

Question 1. The Discrete Time Fourier Transform (DTFT) of a sequence $\{x_k\}$, $k = 0, \pm1, \pm2 \ldots, \pm\infty$ is given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-jk\omega}. \tag{1}$$

Question 1a Explain why the DTFT is not computable on a digital computer.
Answer. It is impossible to implement because
- the sum is over an infinite number of samples.
- the frequency ω is continuous.

Question 1b Consider now the Discrete Fourier Transform (DFT) of $\{x_k\}$ given for $n = 0, \ldots, N-1$ by

$$X_n = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi kn}{N}}. \tag{2}$$

This DFT only provides N spectrum values, whis is insufficient in most applications. Show how it is possible to compute

$$X'(\omega) = \sum_{k=0}^{N-1} x_k e^{-jk\omega}$$

as a function of $\{X_n\}$.

In practice, we are often only interested in computing $X'(\omega)$ at the discrete frequencies $\frac{2\pi n}{P}$ where $n = 0, 1, \ldots, P-1$ and $P > N$. Name and describe an efficient way based on the DFT to compute these values.

Answer. The Inverse Discrete Fourier Transform yields

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{j\frac{2\pi nk}{N}}.$$

Hence, one has

$$\begin{aligned}
X'(\omega) &= \sum_{k=0}^{N-1} x_k e^{-jk\omega} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} X_n e^{-jk\omega} e^{j\frac{2\pi nk}{N}} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} X_n \left(\sum_{k=0}^{N-1} e^{-jk\left(\omega - \frac{2\pi n}{N}\right)}\right) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} X_n \frac{1 - e^{-jN\left(\omega - \frac{2\pi n}{N}\right)}}{1 - e^{-j\left(\omega - \frac{2\pi n}{N}\right)}} \qquad \leftarrow \text{if } \omega \neq \frac{2\pi n}{N} \text{ for } n \in \{0, N\text{-} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} X_n e^{-j\left(\frac{N-1}{2}\right)\left(\omega - \frac{2\pi n}{N}\right)} \frac{\sin\left(\frac{N}{2}\left(\omega - \frac{2\pi n}{N}\right)\right)}{\sin\left(\frac{1}{2}\left(\omega - \frac{2\pi n}{N}\right)\right)}.
\end{aligned}$$

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In practice, one does not use this formula but the zero-padding technique. A sequence of length P is created, the N first elements correspond to $\{x_k\}$ and the remaining ones are set equal to zero. Then the DFT of this P-length sequence is computed.

Question 1c Assume you are interested in computing the DFTs of two real-valued sequences $\{x_k\}$ and $\{y_k\}$ of length N ($k = 0, 1, \ldots, N-1$). Show how it is possible to compute these DFTs using the DFT of a single complex-valued signal $\{z_k\}$ of length N given by

$$z_k = x_k + jy_k.$$

How would this algorithm be implemented practically?

Answer. We know that the DFT is linear, thus

$$Z_n = X_n + jY_n.$$

Moreover we have

$$x_k = \mathrm{Re}\,(z_k) = \frac{z_k + z_k^*}{2},$$
$$y_k = \mathrm{Im}\,(y_k) = \frac{z_k - z_k^*}{2j}.$$

We can check easily that the DFT of the conjugate sequence $\{z_k^*\}$ is equal at frequency $\frac{2\pi n}{N}$ to

$$Z_{N-n}^*$$

thus

$$X_n = \frac{Z_n + Z_{N-n}^*}{2}$$
$$Y_n = \frac{Z_k - Z_{N-n}^*}{2j}$$

In practice, the DFT is implemented using the FFT algorithm.

Question 2.

Question 2a [15%] Describe the backward difference and bilinear transform methods to convert analog filters to digital filters. Explain why the bilinear transform is usually favoured.

Answer. In the backward difference method, we set simply

$$s = T^{-1}\left(1 - z^{-1}\right) \Leftrightarrow z = (1 - sT)^{-1}$$

where T is the sampling period. The problem with this approach is that the left half plane is mapped into a circle of centre $(0.5, 0)$ and radius 0.5. This mapping has the property that a stable analog filter remains stable in the digital domain but the poles are confined to a relatively small set of frequencies. It is impossible to design highpass filters with such an approach.

The bilinear transform is given by

$$s = k\frac{1 - z^{-1}}{1 + z^{-1}}.$$

This transform does map the left half plane into the unit circle. In particular it allows the user to design highpass filters. The main problem with this approach is it performs a nonlinear mapping of the phase leading to a distortion of the digital frequency response. This effect can be compensated partially by prewarping the analogue filter before applying the bilinear transformation.

2 (b) (i) Autocorrelation

$$r_{XX}[k_1, k_2] = E[X_{k_1} X_{k_2}]$$

Cross-correlation:

$$r_{XY}[k_1, k_2] = E[X_{k_1} Y_{k_2}]$$

(ii) <u>WSS</u>

Required mean to be const.

$$\mu = E[X] = \text{const.}$$

Also variance ~~$r_{XX}[k_1, k_2]$~~ is finite and

$r_{XX}[k_1, k_2]$ depends only on lag $k_2 - k_1$:

$$r_{XX}[k_1, k_2] \to r_{XX}[k_2 - k_1]$$

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Question 2c [35%] A highpass filter is required with sampling rate 32.20khz. The 3dB corner frequency is 6.50kHz. We are interested in designing this infinite impulse response (IIR) digital filter from an analogue prototype given by

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Using the lowpass to highpass transformation

$$s \longrightarrow \frac{\Omega_c}{s},$$

where Ω_c is the corner frequency, together with the bilinear transform, design the required digital filter. Determine the zeros and poles of this digital filter.

Answer. The normalized frequency is given by

$\omega_c = 1.2683$

$$\omega_c = 2\pi \frac{6.50}{32.20} \Rightarrow \Omega_c = \tan\left(\frac{\omega_c}{2}\right) = 0.736.$$

Combining this and the bilinear transform, we get first the analog highpass filter

$$H(s) = \frac{1}{\left(\frac{\Omega_c}{s}\right)^2 + \sqrt{2}\left(\frac{\Omega_c}{s}\right) + 1} = \frac{s^2}{s^2 + \sqrt{2}\Omega_c s + \Omega_c^2}$$

and using

$$s \longrightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

then

$$\begin{aligned}
H(z) &= \frac{1}{\Omega_c^2 \left(\frac{1+z^{-1}}{1-z^{-1}}\right)^2 + \sqrt{2}\Omega_c \left(\frac{1+z^{-1}}{1-z^{-1}}\right) + 1} \\
&= \frac{\left(1 - z^{-1}\right)^2}{\Omega_c^2 \left(1 + z^{-1}\right)^2 + \sqrt{2}\Omega_c \left(1 - z^{-2}\right) + \left(1 - z^{-1}\right)^2} \\
&= \frac{\left(1 - z^{-1}\right)^2}{\cancel{\left(\Omega_c^2 - \sqrt{2}\Omega_c\right) z^{-2} + \left(2\Omega_c^2 - 1\right) z^{-1} + \Omega_c^2 + \sqrt{2}\Omega_c + 1}} \\
&= \frac{\left(1 - z^{-1}\right)^2}{\cancel{-0.499 z^{-2} + 0.082 z^{-1} + 2.58}} \\
&= \frac{-2\left(1 - z^{-1}\right)^2}{\cancel{\left(2.3575 - z^{-1}\right)\left(-2.1932 - z^{-1}\right)}}
\end{aligned}$$

$$= \frac{(1 - z^{-1})^2}{(\Omega_c^2 - \sqrt{2}\Omega_c + 1) z^{-2} + (2\Omega_c^2 - 2) z^{-1} + (\Omega_c^2 + \sqrt{2}\Omega_c + 1)}$$

$$= \frac{(1 - z^{-1})^2}{2.5812 - 0.918 z^{-1} + 0.5008 z^{-2}}$$

Hence the digital filter has a zero at 1 and poles at $1/2.3575 = 0.424$ and $-1/2.193 = -0.46$.

$$= \frac{0.3874 - 0.7748 z^{-1} + 0.3874 z^{-2}}{1 - 0.3556 z^{-1} + 0.194 z^{-2}}$$

2 (d) An FIR filter has transfer function

$$H(z) = (1 - z^{-1})^2$$

A white noise process having variance equal to 1 is placed at the input to the filter.

Determine the cross-correlation function between the white noise input process and the filtered output.

Determine also the power at the output

$$r_{xx} = \delta[n] \times 1$$

$$r_{xy} = h_p * r_{xx}$$

$$= \begin{array}{ccc} [\,1 & -2 & 1\,] \\ r_{xy}[0] & r_{xy}(1) & r_{xy}[2] \end{array}$$

$$\left(H(z) = 1 - 2z^{-1} + z^{-2} \right)$$

$$r_{yy}[0] = 1 . \sum h_p^2 = 6.$$

3 (a)

(i) $J = E[\varepsilon_n^2] = E[(x_n - \hat{x}_n)^2]$

$$\frac{\partial J}{\partial h_q} = E\left[2\varepsilon_n \frac{\partial \varepsilon_n}{\partial h_q}\right]$$

$$\frac{\partial \varepsilon_n}{\partial h_q} = -y_{n-q}$$

$$\Rightarrow \frac{\partial J}{\partial h_q} = -2E\left[\left(x_n - \sum_p h_p y_{n-p}\right) y_{n-q}\right]$$

$$= -2 r_{xy}[-q] + 2\sum_p h_p r_{yy}[q-$$

$$= -2 r_{yx}[q] + 2\sum_p h_p r_{yy}[q-p]$$

$\Rightarrow$ At min J:

$$\sum h_p r_{yy}[q-p] = r_{yx}[q],$$

$$-\infty < q < \infty$$

As required.

3.(b) (ii)

$$r_{yx}[p] = E[(x_n + 0.4\, x_{n-1} + V_n)\, x_{n+p}]$$

$$= r_{xx}[p] + 0.4\, r_{xx}[p+1] + \cancel{r_{vx}[p]}^{\,0}$$

$$= 0.9^{|p|} + 0.4 \times 0.9^{|p+1|}$$

$$r_{yy}[p] = E[y_n(x_{n+p} + 0.4\, x_{n+p-1} + V_{n+p})]$$

$$= r_{yx}[p] + 0.4\, r_{yx}[p-1] + r_{yv}[p]$$

$$= 0.9^{|p|} + 0.4 \times 0.9^{|p+1|} + 0.4 \times (0.9^{|p-1|} + 0.4 \times 0.9^{|p|}) + r_{vv}[p]$$

$$r_{yv}[p] = E[(x_n + 0.4\, x_{n+1} + V_n)\, V_{n+p}] = r_{vv}[p]$$

since v & x uncorrelated

3(b) ~~(ii)~~ contd.

$$r_{YY}[p] = (1+0.16)\,0.9^{|p|} + 0.4\,\big(0.9^{|p-1|} + 0.9^{|p+1|}\big) + \delta_p$$

3(c)

Frequency Domain Wiener filter:

Take DTFT of

$$\sum_p h_p\, r_{YY}[q-p] = r_{YX}[q]$$

$$\rightarrow \quad H(e^{j\theta})\, S_Y(e^{j\theta}) = S_{YX}(e^{j\theta})$$

Now,

$$S_{YX}(e^{j\theta}) = \text{DTFT}\{0.9^{|p|} - 0.4\times 0.9^{|p+1|}$$

$$= S_X(e^{j\theta}) + 0.4\times \underbrace{e^{j\theta} S_X(e^{j\theta})}_{\text{'time delay' of}}$$

$$\& \quad S_{YY}(e^{j\theta}) = \text{DTFT}\{1.16\, r_{XX}[p] + 0.4\,\big(r_{XX}[p-1] + r_{XX}[p+1]\big) - \delta_p\}$$

$$= 1.16\, S_X(e^{j\theta}) + 0.4\,(e^{j\theta} + e^{-j\theta})\, S_X(e^{j\theta})$$

$+ \; 1$

3(c) ~~(ii)~~ Contd.

$$\Rightarrow \quad H(e^{j\theta}) = \frac{S_{YX}(e^{j\theta})}{S_{YY}(e^{j\theta})}$$

$$= \frac{1 + 0.4\,e^{j\theta}}{1.16 + 0.8\cos\theta + \frac{1}{S_X(e^{j\theta})}}$$

But $\quad S_X(e^{j\theta}) = \dfrac{1 - 0.9^2}{|1 - 0.9\,e^{-j\theta}|^2}$

by given result.

$$\Rightarrow \quad H(e^{j\theta}) = \frac{1 + 0.4\,e^{j\theta}}{1.16 + 0.8\cos\theta + \frac{1}{0.19}|1 - 0.9e^{-j\theta}|^2}$$

$1/0.81$

Solutions to 3F3 Pattern Processing Questions, 2005

1. *Parameter Estimation*

   (a) Answer should include:

   - need to decide on nature/form of the class-conditional distributions;
   - mention of generalisation i.e. enough training data to robustly estimate the model parameters;

   Bayes' decision rule is optimal when:

   - the form of class-conditional distributions and prior are known;
   - infinite training data is available;
   - the global maximum is found.

   Since these conditions are rarely satisfied it is usually reasonable to use these alternative approaches.

   (b) Form of log-likelihood

$$\begin{aligned}\log(p(\mathbf{X}|\mu)) &= \sum_{i=1}^{N}\log(\mathcal{N}(x_i;\mu,\sigma^2)) \\ &= \sum_{i=1}^{N}\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{(x_i-\mu)^2}{2\sigma^2}\end{aligned}$$

   The ML estimate of the mean will be given by differentiating wrt μ and equating to zero. Thus

$$\frac{\partial}{\partial\mu}\log(p(\mathbf{X}|\mu)) = \sum_{i=1}^{N}\frac{(x_i-\mu)}{\sigma^2}$$

   Equating to zero yields

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N}x_i$$

   (c)(i) The value of the mean will be 1.

   (c)(ii) The Bayes decision boundary will occur when the two log-likelihoods are the same (equal priors). Thus a point x on the decision boundary will satisfy

$$\log\left(\frac{1}{\sqrt{2\pi\times10^6}}\right) - \frac{(x-1)^2}{2\times10^6} = \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{x^2}{2}$$

Thus

$$x^2 - \frac{(x-1)^2}{10^6} = \log(10^6)$$

(c)(iii) For part (c)(ii) the decision boundary needs to occur at about $\sqrt{\log(10^6)}$. Since the assumed variance of the PDF for class ω_2 is 1, the same as that of class ω_1, the decision boundary will occur half way between the two means. This needs to be close to $\sqrt{\log(10^6)}$, so an estimate of μ which is closer to the Bayes decision (and hence yield a lower error rate is

$$\mu = 2\sqrt{\log(10^6)} \qquad \simeq 7.4338$$

A more accurate estimate is possible by taking into account the second term, this will yield a slightly larger value (with a fractionally lower error rate).

In practice the question is worded so that provided that a reasoned argument is used an increase in the value of μ is acceptable.