UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

Tuesday 10 May 2005 9 to 10.30

Module 3C5

DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 7

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1 The motion of a rigid body is described by its linear momentum $\boldsymbol{p}$ and its moment of momentum $\boldsymbol{h}_P$. The body is subject to an external force $\boldsymbol{F}^{(e)}$ and to an external couple $\boldsymbol{Q}_P^{(e)}$. Moments are taken about a general moving point P whose motion is described by the position vector $\boldsymbol{r}_P$.

(a) Beginning with Newton's laws for a particle, derive the data sheet results for the motion of a rigid body:

(i) $\boldsymbol{F}^{(e)} = \dot{\boldsymbol{p}}$ and [25%]

(ii) $\boldsymbol{Q}_P^{(e)} = \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$. [50%]

(b) Show that a special result holds when P coincides at all times with the centre of mass of the body. [25%]

2 (a) A uniform circular plate of mass m and radius a lies in the x-y plane and has its centre at the origin O. Two particles, each of mass $m/2$, are attached to the plate at $(0.6a, 0.8a, 0)$ and $(-0.6a, -0.8a, 0)$. Find:

(i) the inertia matrix at O referred to the x-y-z reference frame; [25%]

(ii) the principal moments of inertia and the directions of the principal axes. [25%]

(b) A solid cube of mass m has edges of length $2a$. The cube is centred at O with its faces parallel with the x-y, y-z and x-z planes. Two particles, each of mass $m/2$, are attached to the cube at $(a, a/2, a/2)$ and $(-a, -a/2, -a/2)$.

Find the principal moments of inertia and the directions of the principal axes. [50%]

3 A rotor of mass m and radius a spins freely on a shaft BG. The centre of mass of the rotor is at G. The shaft is fixed at B to a light string OB of length b as shown in Fig. 1. The string is fixed to the ceiling at O. The distance BG is equal to the rotor radius a and BG is aligned with the axis of the rotor. The rotor is spinning at a constant *fast* rate ω and its polar moment of inertia can be taken as $ma^2/2$.

In steady-state precession the angle α between BG and the horizontal and the angle β between OB and the vertical are both constant as shown in the figure. The plane OBG remains vertical and rotates at a steady rate Ω.

(a) By drawing a free-body diagram of the shaft and rotor:

(i) find the tension in the string in terms of m, g and β; [20%]

(ii) show that the string angle $\beta \approx \dfrac{a\Omega^2}{g - b\Omega^2}$ when α and β are small. [20%]

(b) In a particular demonstration the string length b is chosen to be equal to g/Ω^2. What might be observed in such a demonstration? Show how you would compute the string angle β without making small-angle approximations. [30%]

(c) Use the gyroscope equations to find an expression for the precession rate Ω in terms of g, a and ω. [30%]

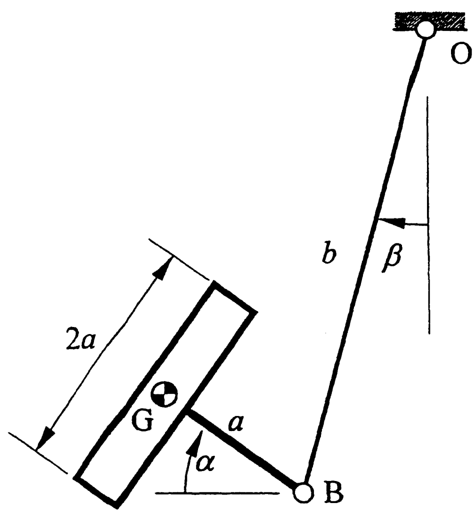

Fig. 1

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4 A vehicle of mass m is free to roll without friction on a horizontal surface. A tall mast, also of mass m, is fixed to the vehicle through a frictionless pivot at A as shown in Fig. 2. The mast may be modelled as a uniform rigid rod of length $2a$. A torsional spring of stiffness k is fitted at A between the mast and the vehicle. The horizontal position of the vehicle is x and the angle of tilt of the mast from the vertical is θ as shown in the figure. The spring is chosen to be sufficiently stiff so that there is a stable equilibrium at $\theta = 0$.

(a) Find expressions for the kinetic energy and potential energy of the vehicle and its mast. [30%]

(b) For small vibration of the vehicle and mast about their equilibrium positions, find the mass and stiffness matrices. [30%]

(c) Find the two natural frequencies of small vibration and the corresponding mode shapes. [30%]

(d) What is the minimum value of the spring stiffness k required to achieve a stable equilibrium at $\theta = 0$? [10%]

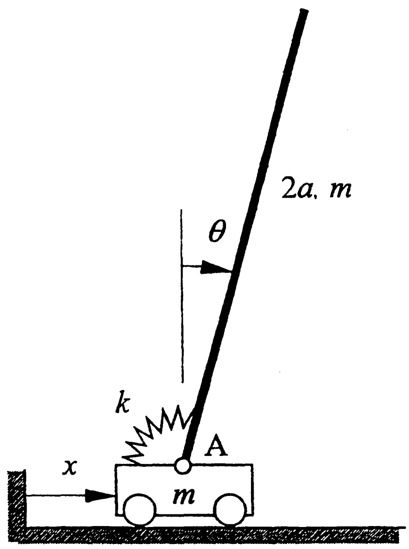

Fig. 2

5 A wedge-shaped trolley of mass m, shown in Fig. 3, is constrained to move without friction on the surface of a horizontal table. A uniform solid cylinder, also of mass m, is free to roll without slip on the inclined surface of the wedge. The radius of the cylinder is r and the angle of the wedge is α as shown in the figure. A horizontal force f is applied to the trolley as shown. Generalized coordinates used to describe the motion of the trolley and the cylinder are x, the position of the trolley, and θ, the rotation of the cylinder. Do not assume that x and θ are small.

(a) What are the generalized forces corresponding to coordinates x and θ? [10%]

(b) Find expressions for the potential and kinetic energy of the system. [30%]

(c) Use Lagrange's equation to obtain equations of motion for the system. [30%]

(d) Use your equations to determine:

(i) the force f required to keep θ constant; [10%]

(ii) the motion of the trolley in the absence of any applied force f when the cylinder and trolley are released from rest. [20%]

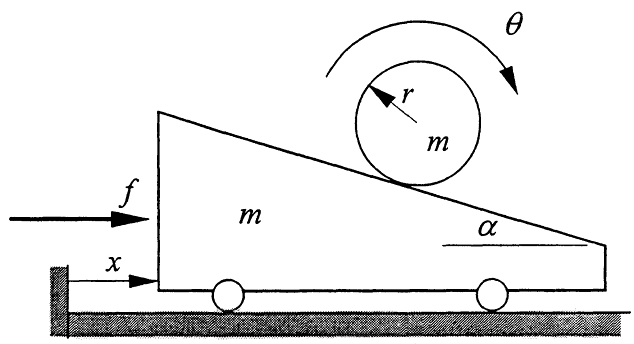

Fig. 3

END OF PAPER

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Part IIA Data sheet **S32**

Module 3C5 Dynamics

Module 3C6 Vibration

Dynamics in three dimensions

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity ω about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, dm \;=\; \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$$
$$D = \int yz\,dm \qquad E = \int zx\,dm \qquad F = \int xy\,dm$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\bar{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\bar{T}}$ is *stationary* near each mode.)

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VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^{\,2} M\underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

2. Kinetic energy

$$T = \frac{1}{2}\underline{\dot{u}}^t M \underline{\dot{u}}$$

3. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\,\underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and a_n gives the "amount" of the nth mode.

5. Transfer function

For (generalised) force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^{\,2} - \omega^2}$$

(with no damping), or

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

2. Kinetic energy

$$T = \frac{1}{2}\int \dot{u}^2\, dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Orthogonality and normalisation

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\, u_n(x)$$

where $w(x,t)$ is the displacement and a_n gives the "amount" of the nth mode.

5. Transfer function

For force F at frequency ω applied at point x, and response w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^{\,2} - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{q}{F} \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

7. Impulse response

For a unit impulse applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$g(j,k,t) = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(j,k,t) \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

8. Step response

For a unit step force applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$h(j,k,t) = \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(j,k,t) \approx \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(t) \approx \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

PART IIA 2005

3C5 Dynamics

Principal Assessor: **Dr H E M Hunt**

Answers

1. (b) $\underline{Q}_p^{(e)} = \dot{\underline{h}}_p$

2. (a) (i) $\begin{bmatrix} 0.89 & -0.48 & 0 \\ -0.48 & 0.61 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} ma^2$

(ii) $0.25ma^2$, $1.25ma^2$, $1.5ma^2$, directions aligned with particles, perpendicular to line of particles and perpendicular to plate respectively.

(b) AAC = $13ma^2/6$, $13ma^2/6$, $2ma^2/3$, directions of A are anywhere perpendicular to the line of particles, and C is aligned with the line of particles.

3. (a) (i) $mg/\cos\beta$

(b) steady-state β close to $\pi/2$

(c) $2g/a\omega$ for fast spin

4. (a) $T = m\dot{x}^2 + ma\dot{x}\dot{\theta}\cos\theta + \frac{2}{3}ma^2\dot{\theta}^2$ $\qquad V = \frac{1}{2}k\theta^2 + mga(\cos\theta - 1)$

(b) $M = \begin{bmatrix} 2m & ma \\ ma & \frac{4}{3}ma^2 \end{bmatrix}$ $\qquad K = \begin{bmatrix} 0 & 0 \\ 0 & k - mga \end{bmatrix}$

(c) $\omega = 0$, rigid body translation, $\{x,\theta\} = \{1,0\}$

$\omega = \frac{6}{5ma^2}(k - mga)$, modeshape $\{x,\theta\} = \{a, -2\}$

(d) $k > mga$

5. (a) $Q_x = f \qquad Q_\theta = 0$

(b) $T = m\dot{x}^2 + ma\dot{x}\dot{\theta}\cos\alpha + \frac{3}{4}ma^2\dot{\theta}^2$ $\qquad V = -mga\theta\sin\alpha$

(c) $2m\ddot{x} + ma\ddot{\theta}\cos\alpha = f$ $\qquad \ddot{x}\cos\alpha + \frac{3}{2}a\ddot{\theta} = g\sin\alpha$

(d) (i) $2mg\tan\alpha$ $\qquad$ (ii) $\ddot{x} = \frac{g\sin\alpha\cos\alpha}{\cos^2\alpha - 3}$

HEMH May 2005