UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

Friday 13 May 9.00 to 10.30

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 3

(TURN OVER

1 A vertical elastic column of length L, cross-sectional area A, Young's modulus E and density ρ is supported at its base on a spring of stiffness K, which rests on a rigid foundation as shown in Fig. 1. The top of the column is unconstrained. The axial displacement at distance x from the base and time t is $w(x, t)$.

(a) Write down the differential equation governing free axial vibration of the column, and give the boundary conditions appropriate to the two ends of the column. [25%]

(b) Show that the natural frequencies ω are the roots of the equation

$$\tan\frac{\omega L}{c} = \frac{Kc}{EA\omega}$$

where c is a constant that you should define. [25%]

(c) Sketch a graphical solution of this equation, and sketch the mode shapes corresponding to the first three natural frequencies. [25%]

(d) Discuss what happens to the solution from part (c) in the limits
(i) $K \to 0$
(ii) $K \to \infty$
and explain the physical interpretation of these limiting cases. [25%]

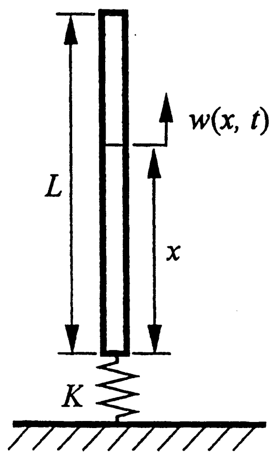

Fig. 1

2 A cantilever beam has length L, bending rigidity EI and mass per unit length m.

(a) Use dimensional analysis to show that any particular natural frequency ω_n must satisfy

$$\omega_n = \frac{\lambda_n}{L^2}\sqrt{\frac{EI}{m}}$$

where λ_n is a dimensionless constant. [30%]

(b) Write down the differential equation and boundary conditions relevant to bending vibration of the cantilever beam. Outline the method by which these equations can be used to find the natural frequencies. Fully detailed calculations are not needed but you should explain enough to show that the result is consistent with the answer to part (a). [30%]

(c) Sketch the first three mode shapes of the beam. [10%]

(d) Two identical beams of the kind investigated above are connected together by a rigid, massless link joining the tips of the two beams, as illustrated in Fig. 2. The link is freely hinged to each beam. Without detailed calculations, sketch the first four mode shapes of this coupled system in the order of their natural frequencies, assuming that the vibration of the beams is confined to the plane of the diagram. Explain briefly how this frequency order can be justified. [30%]

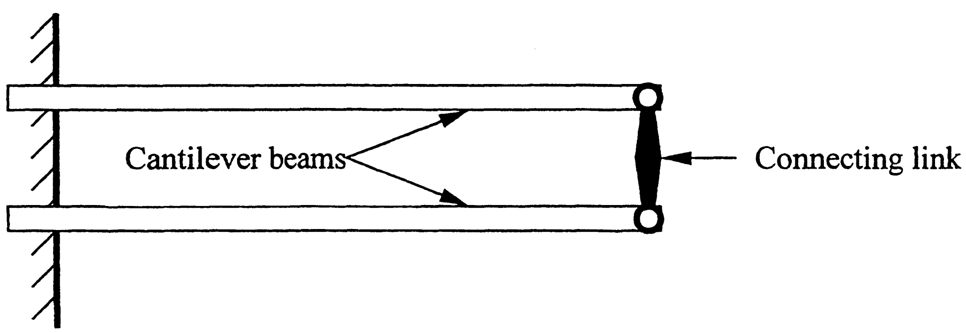

Fig. 2

(TURN OVER

3 Three masses m, m and M are connected together and to a fixed support by four springs of stiffness k as shown in Fig. 3. Each mass can move in the horizontal direction only, without rotation. The displacements from equilibrium are denoted x_1, x_2 and y as shown.

(a) Write an expression for the kinetic energy and show that the potential energy is

$$V = k\left(x_1^2 + x_2^2 + y^2 - x_1 y - x_2 y\right).$$ [20%]

(b) Sketch the mode shapes and write down estimates of the natural frequencies for the cases:

(i) $m/M \ll 1$

(ii) $m/M \gg 1$.

For each case state which one of these frequencies is exact. [40%]

(c) For the case in (b)(i) use Rayleigh's quotient with the mode shape $(x_1, x_2, y)^T = (1, 1, \alpha)^T$ to find an exact expression for the remaining two natural frequencies. Confirm that these frequencies are consistent with your estimates in part (b). [40%]

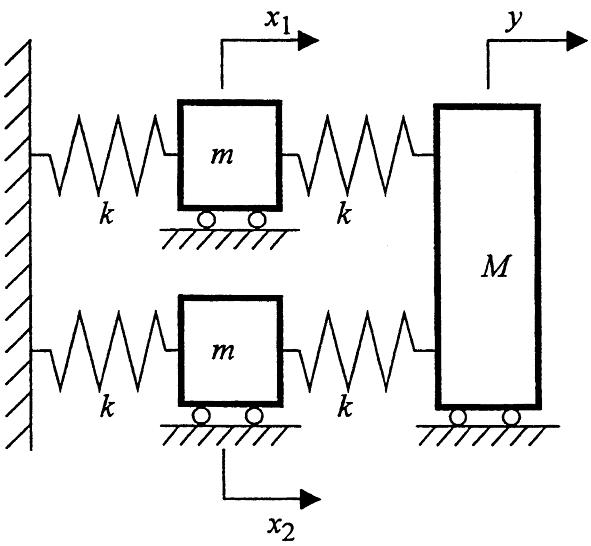

Fig. 3

4. Vibration of a three-wheeled 'economy' vehicle can be modelled as shown in Fig. 4. A light, rigid, triangular frame of length $4a$ and width $4b$ is supported at its corners by three identical springs of stiffness k, resting on a flat, rigid road surface. The mass distribution can be modelled by three identical lumped masses m, located at the positions shown. Small vibration of the system is described by the three generalised coordinates z, θ, ϕ which are the vertical displacement, 'roll' and 'pitch' rotations about the mid-lines of the frame.

(a) Derive expressions for the potential and kinetic energies, about the position of static equilibrium and hence show that the mass and stiffness matrices are

$$[M] = m\begin{bmatrix} 3 & 0 & -a \\ 0 & 2b^2 & 0 \\ -a & 0 & 3a^2 \end{bmatrix}\begin{matrix} z \\ \theta \\ \phi \end{matrix} \quad \text{and} \quad [K] = k\begin{bmatrix} 3 & 0 & -2a \\ 0 & 8b^2 & 0 \\ -2a & 0 & 12a^2 \end{bmatrix}\begin{matrix} z \\ \theta \\ \phi \end{matrix}.$$ [30%]

(b) One vibration mode can be deduced without calculation. Describe it and give its natural frequency. Find the remaining two vibration modes and their natural frequencies. Sketch these two mode shapes and show the nodal points. [50%]

(c) Without further calculations, sketch the amplitude (on a dB scale) of the transfer function for a displacement at the point B in response to a vertical harmonic force applied to point A. [20%]

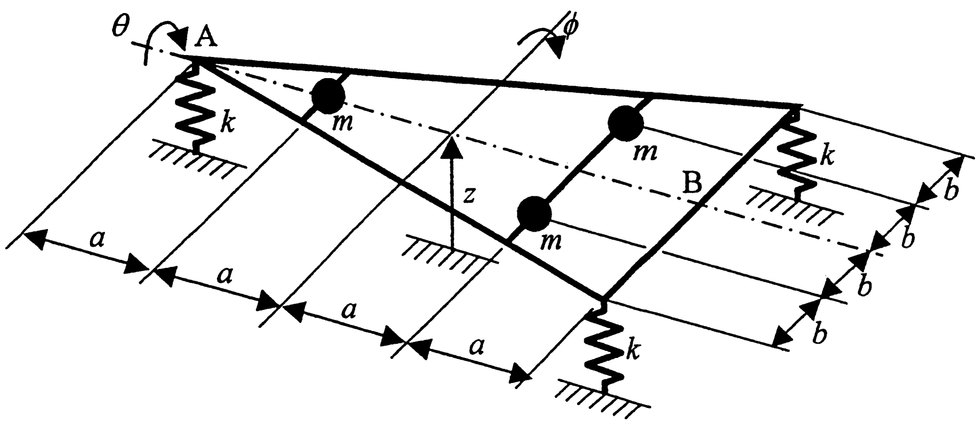

Fig. 4

END OF PAPER

UNIVERSITY OF
CAMBRIDGE

UNIVERSITY OF CAMBRIDGE

Part IIA Data sheet **S32**

Module 3C5 Dynamics

Module 3C6 Vibration

Dynamics in three dimensions

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{dp}{dt} = F^{(e)}$$

where $p = M\,v_G$, M is the total mass, v_G is the velocity of the centre of mass and $F^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$Q^{(e)} = (r_G - r_P) \times \dot{p} + \dot{h}_G$$
$$= \dot{h}_P + \dot{r}_P \times p$$

where $Q^{(e)}$ is the total moment of external forces about P. Here, h_P and h_G are the moments of momentum about P and G respectively, so that for example

$$h_P = \sum_i (r_i - r_P) \times m_i \dot{r}_i$$
$$= h_G + (r_G - r_P) \times p$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity ω about a fixed point P at the origin of coordinates

$$h_P = \int r \times (\omega \times r)\, dm = I\,\omega$$

where the integral is taken over the volume of the body, and where

$$I = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2) dm \qquad B = \int (z^2 + x^2) dm \qquad C = \int (x^2 + y^2) dm$

$D = \int yz\, dm \qquad E = \int zx\, dm \qquad F = \int xy\, dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{p} + \Omega \times p = F^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector r is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\bar{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\bar{T}}$ is *stationary* near each mode.)

UNIVERSITY OF CAMBRIDGE

VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^{\,2} M\underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

2. Kinetic energy

$$T = \frac{1}{2}\underline{\dot{u}}^t M \underline{\dot{u}}$$

3. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\, \underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and a_n gives the "amount" of the nth mode.

5. Transfer function

For (generalised) force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^{\,2} - \omega^2}$$

(with no damping), or

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

$$T = \frac{1}{2}\int \dot{u}^2 \, dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\, u_n(x)$$

where $w(x,t)$ is the displacement and a_n gives the "amount" of the nth mode.

For force F at frequency ω applied at point x, and response w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^{\,2} - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{q}{F} \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

7. Impulse response

For a unit impulse applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$g(j,k,t) = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(j,k,t) \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

8. Step response

For a unit step force applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$h(j,k,t) = \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(j,k,t) \approx \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(t) \approx \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.