ENGINEERING TRIPOS PART IIA

Thursday 12 May 2005 9.00 to 10.30

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheet (2 pages).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 A thin square plate of side length a (Fig. 1) is subjected to uniform tension $\sigma_{xx} = \sigma$ in the x-direction and uniform compression $\sigma_{yy} = -\sigma$ in the y-direction. As a result, the plate elongates by an amount Δa in the x-direction, contracts by Δa in the y-direction whilst its diagonal rotates by an angle $\gamma/2$. The plate is made of an isotropic material having Young's modulus E, Poisson's ratio ν and shear modulus G.

(a) Establish the elastic relationship between $\Delta a / a$ and σ. [20%]

(b) Show that the shear stress τ acting on the diagonal plane of the plate is equal to σ. [15%]

(c) Define the engineering shear strain, and establish the elastic relationship between τ and γ. [15%]

(d) Derive the geometrical relationship between $\Delta a / a$ and γ, assuming $\Delta a / a \ll 1$ and $\gamma \ll 1$. [30%]

(e) Use the above relationships to show that

$$G = \frac{E}{2(1+\nu)}.$$ [10%]

(f) In addition to E, ν and G, list one more material parameter for an isotropic elastic material. How many independent material properties does an isotopic elastic material have? [10%]

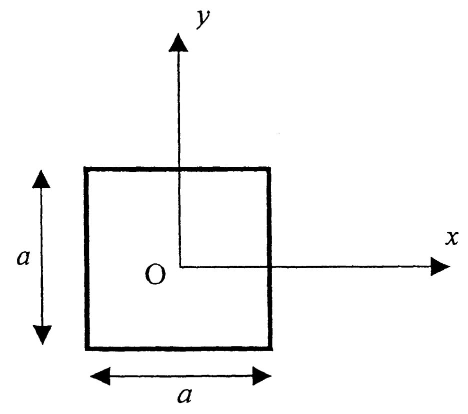

Fig. 1

2 A long cylinder has inner radius a and outer radius b. The cylinder is made of an isotropic material having Young's modulus E and Poisson's ratio ν. You may assume plane strain conditions in this situation.

(a) If the external loading on the cylinder has circular symmetry, use the Lamé solution to show that the radial displacement u at a radius r within the cylinder can be expressed as

$$u = \frac{r(1+\nu)}{E}\left[A(1-2\nu) + \frac{B}{r^2}\right]$$

where A and B are constants. [40%]

(b) The cylinder is now buried underground and subjected to internal pressure p_0. The pressure p_e exerted on the outer surface of the cylinder by the surroundings has been approximated as

$$p_e = (E/b)\left[u\right]_{r=b}.$$

(i) Using the boundary conditions at $r = a$ and $r = b$, express p_e in terms of p_0, E and ν. [40%]

(ii) Calculate the change in the thickness of the cylinder wall. [20%]

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3 Figure 2 shows an elastic cantilever OAB in the form of a triangular plate. The thickness t of the plate may be assumed to be much smaller than any other dimension. Side OA is horizontal, side AB is built into a rigid support and the angle $\mathrm{AOB} = \alpha$. Side OA carries a uniformly distributed pressure of magnitude p. Employing polar coordinates with the origin at O and the angle θ measured downwards from OA, a suitable Airy stress function for this problem is

$$\phi = \frac{Cr^2}{\tan\alpha - \alpha}\left[\alpha - \theta + \frac{\sin 2\theta}{2} - \tan\alpha \cos^2\theta\right]$$

where C is a constant.

(a) Determine the stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}$ at any point in the cantilever. [30%]

(b) State the boundary conditions along OA and OB and show that these boundary conditions are satisfied using the suggested stress function. Hence, determine the value of C in terms of the applied pressure p. [30%]

(c) If the angle α is small:

(i) show using simple beam theory, that at point A at the root of the cantilever, the stress $\sigma_{rr} = 3p/\alpha^2$. [20%]

(ii) show that the value of σ_{rr} at A using the above Airy stress function agrees with the simple beam theory prediction. [20%]

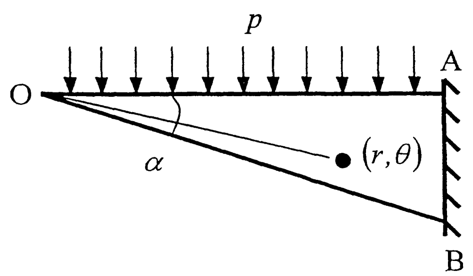

Fig. 2

4 A plane-strain metal forming operation consists of squeezing a billet of width $6l$ by two rigid punches, each of width $4l$ as illustrated in Fig. 3. Both the punches are driven at a speed v towards each other and at some stage of the process are at a distance $x >> l$ apart. All interfaces can be considered to be frictionless and the material behaves as a rigid ideally-plastic Tresca solid with a shear yield strength k. Two possible systems of tangential velocity discontinuities are sketched in Fig. 3 using solid and dashed lines.

(a) Using the system of tangential velocity discontinuities sketched using solid lines, calculate an upper bound to the mean pressure p to bring about continued deformation. [40%]

(b) Using the system of tangential velocity discontinuities sketched using dashed lines, calculate an upper bound to the mean pressure p to bring about continued deformation. Give your answer in terms of x. [40%]

(c) Using (a) and (b) estimate the value of x when the plastic zone first extends from one punch face to the other. [20%]

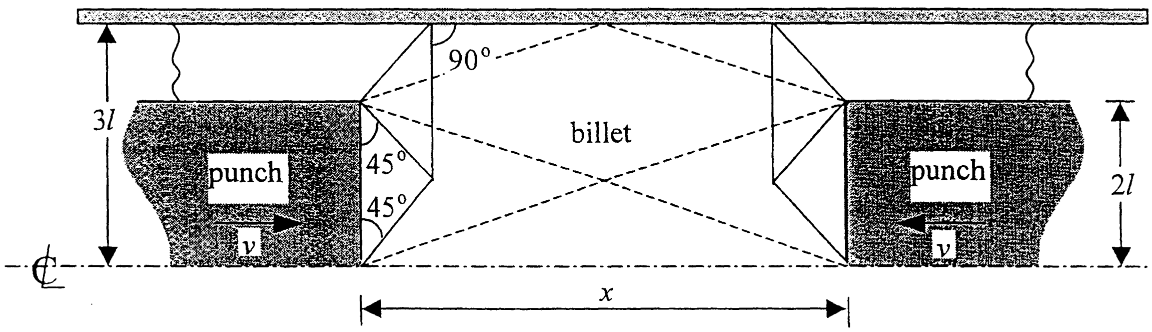

Fig. 3

END OF PAPER

Paper G4: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

Discs and tubes

Equilibrium

$$\sigma_{\theta\theta} = \frac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$$

Lamé's equations (in elasticity)

$$\sigma_{rr} = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\omega^2 r^2 - \frac{E\alpha}{r^2}\int_c^r rT\,dr$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\omega^2 r^2 + \frac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$$

Spheres

Equilibrium

$$\sigma_{\theta\theta} = \frac{1}{2r}\frac{d(r^2\sigma_{rr})}{dr}$$

Lamé's equations (in elasticity)

$$\sigma_{rr} = A - \frac{B}{r^3}$$

$$\sigma_{\theta\theta} = A + \frac{B}{2r^3}$$

2. Plane stress and plane strain

Cartesian coordinates

Strains

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Compatibility

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

or (in elasticity)

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$$

Equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

$\nabla^4\phi = 0$ (in elasticity)

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right\}\left\{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right\} = 0$$

Airy Stress Function

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Polar coordinates

Strains

$$\varepsilon_{rr} = \frac{\partial u}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}$$

Compatibility

$$\frac{\partial}{\partial r}\left\{r\frac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \frac{\partial}{\partial r}\left\{r^2\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$$

or (in elasticity)

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$$

Equilibrium

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$$

$\nabla^4\phi = 0$ (in elasticity)

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right\} \times \left\{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$$

Airy Stress Function

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial \phi}{d\theta}\right\}$$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{dF}{dy}$, $\sigma_{zy}\ (= \tau_y) = -\dfrac{dF}{dx}$

Equilibrium: $T = 2\int_A F\,dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\,\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} .$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\ \{ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\ \{ d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \}^{1/2}$

7. Yield criteria and flow rules

<u>Tresca</u>

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

<u>von Mises</u>

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}} .$$

PART IIA 2005
3C7: Mechanics of solids
Principal Assessor: Dr V Deshpande

Datasheet: Elasticity & Plasticity

Answers to 3C7: Mechanics of Solids (2004-2005)

1. (a) $\dfrac{\Delta a}{a} = \dfrac{1+\nu}{E}\sigma$

   (c) $\tau = G\gamma$

   (d) $\gamma = 2\dfrac{\Delta a}{a}$

   (e) 2 independent parameters

2. (b) $p_e = \dfrac{2\nu p_o(1+\nu)a^2}{b^2[1+(1-2\nu)(1-\nu)]+\nu a^2}$

   (c) $\dfrac{A(1-\nu)}{E}(b-a) + \dfrac{B(1+\nu)}{E}\left(\dfrac{1}{b}-\dfrac{1}{a}\right)$

   where

$$A = \frac{p_o\dfrac{a^2}{b^2} - \dfrac{p_e}{1+\nu}}{(1-2\nu)+\dfrac{a^2}{b^2}}, \quad B = \frac{p_o(1-2\nu) - \dfrac{p_e}{1+\nu}}{\dfrac{(1-2\nu)}{a^2}+\dfrac{1}{b^2}}$$

3. (a)

$$\sigma_{rr} = \frac{C}{\tan\alpha - \alpha}\left(2\alpha - 2\theta - \sin 2\theta - 2\tan\alpha \sin^2\theta\right)$$

$$\sigma_{\theta\theta} = \frac{C}{\tan\alpha - \alpha}\left(2\alpha - 2\theta + \sin 2\theta - 2\tan\alpha \cos^2\theta\right)$$

$$\sigma_{r\theta} = \frac{C}{\tan\alpha - \alpha}\left(1 - \cos 2\theta - \tan\alpha \sin 2\theta\right)$$

   (b) $C = p/2$

4. (a) $p = 6k$

   (b) $p = \dfrac{4k}{xl}\left[l^2 + \dfrac{x^2}{4}\right]$

   (c) $x = 3+\sqrt{5}$