ENGINEERING TRIPOS PART IIA

Saturday 14 May 2005 9 to 10.30

Module 3D7

FINITE ELEMENT METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
Special datasheets (3 pages)

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A planar pin-jointed structure, with the dimensions shown in Fig. 1, consists of eight cable members. Each member behaves linear elastically and has axial stiffness *EA*. The structure must be prestressed to prevent any cable slackening under certain load conditions.

(a) Set up an equilibrium matrix **H** which relates the axial forces in the structure

$$\mathbf{r} = \begin{bmatrix} t_I & t_{II} & t_{III} & t_{IV} & t_V & t_{VI} & t_{VII} & t_{VIII} \end{bmatrix}^T$$

to a set of external loads

$$\mathbf{p} = \begin{bmatrix} p_{EX} & p_{EY} & p_{FX} & p_{FY} & p_{GX} & p_{GY} & p_{HX} & p_{HY} \end{bmatrix}^T$$ [30%]

(b) Verify that

$$\mathbf{r}_0 = \begin{bmatrix} 30 & 0 & 10 & 10 & 0 & 0 & 10\sqrt{2} & 20\sqrt{2} \end{bmatrix}^T$$

is a set of bar forces in equilibrium with the external loads shown in Fig. 1. [10%]

(c) Find a state of self-stress **S** that may exist within the structure. What does the existence of this state of self-stress imply? [20%]

(d) The unloaded structure is initially stress free. Hence find the member forces due to the load shown in Fig. 1. [20%]

(e) Find the minimum initial shortening of member I required to prevent any of the cable tensions being less than 50 N under the given applied loads. [20%]

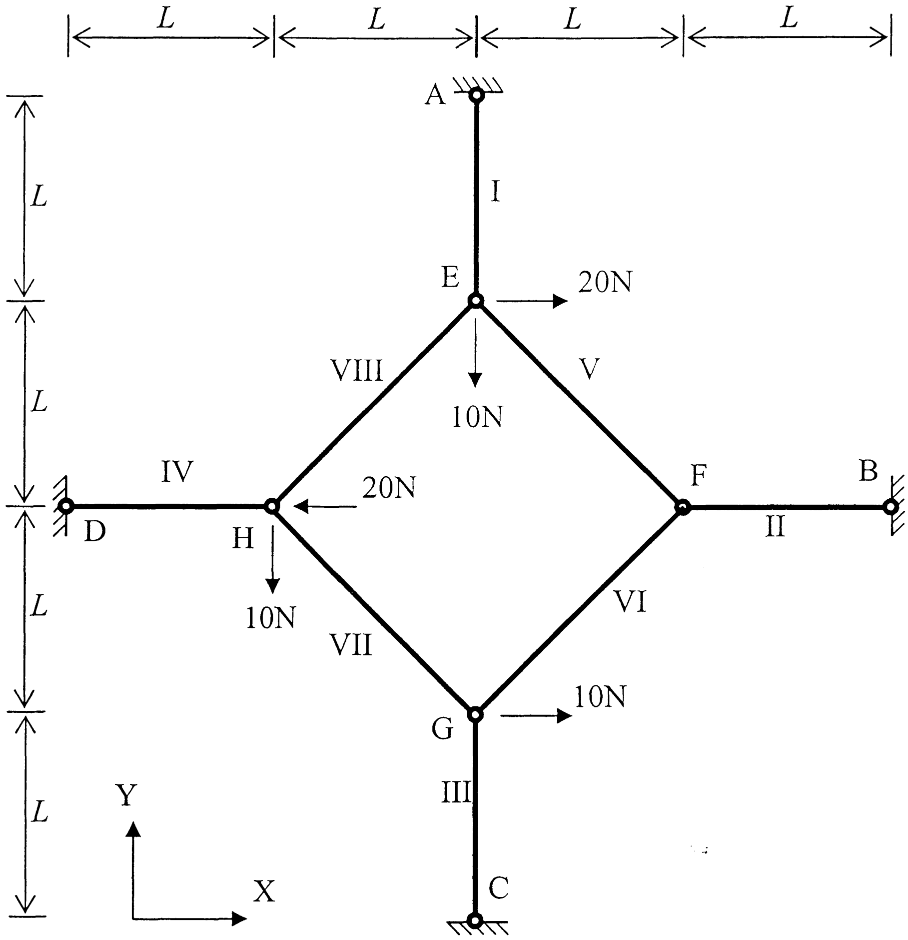

Fig. 1

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2 The stiffness matrix $\mathbf{K}_i'$ of a plane stress, iso-parametric finite element of unit thickness can be obtained from the formula:

$$\mathbf{K}_i' = \int_{\Omega} \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_i \left|\mathbf{J}\right| d\xi d\eta$$

(a) Provide a concise explanation of the meaning of each of the symbols in this formula. [30%]

(b) Fig. 2 shows a four-noded quadrilateral iso-parametric element and the corresponding parent element. Calculate coordinates of the point G in the iso-parametric element that correspond to the Gauss point G' $(1/\sqrt{3}, -1/\sqrt{3})$ in the parent element. [35%]

(c) Evaluate the elements of the matrix **J** at *G* where

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial X}{\partial \xi} & \dfrac{\partial Y}{\partial \xi} \\ \dfrac{\partial X}{\partial \eta} & \dfrac{\partial Y}{\partial \eta} \end{bmatrix}$$

[35%]

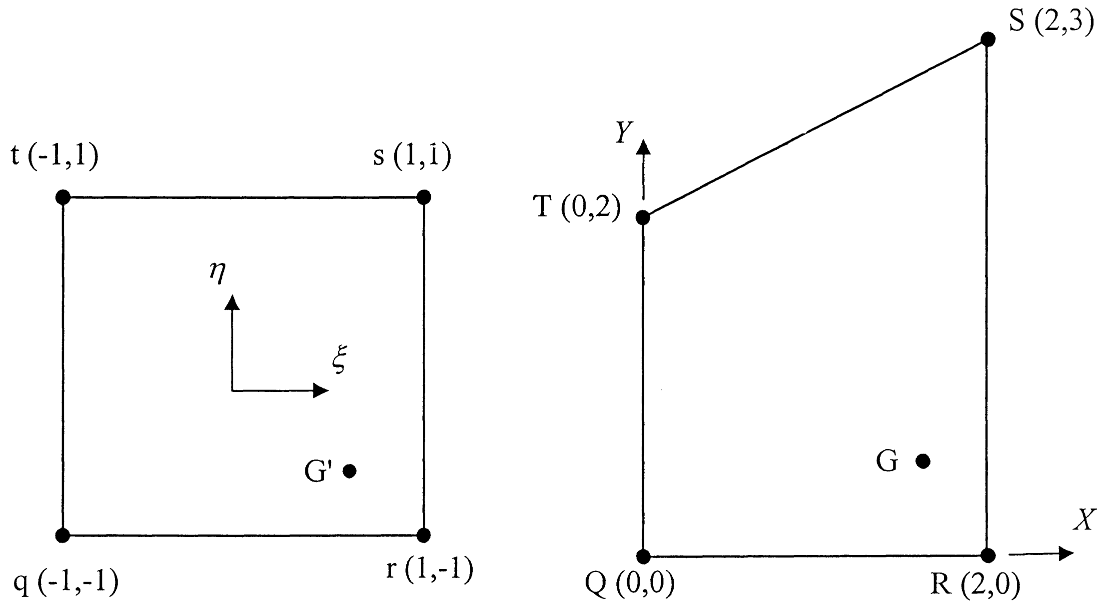

Fig. 2

3 (a) Explain why shape functions n_i for a finite element must satisfy the condition $\sum_i n_i = 1$. [20%]

(b) Construct shape functions n_A to n_D for the four noded triangular element shown in Fig. 3. The shape functions must give a linear variation of displacement on two sides of the triangle and a quadratic variation on the third. [40%]

(c) Nodal displacement components are as follows:

$$\mathbf{d} = \begin{bmatrix} d_{AX} & d_{AY} & d_{BX} & d_{BY} & d_{CX} & d_{CY} & d_{DX} & d_{DY} \end{bmatrix}^T$$
$$= \frac{1}{10}\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}^T$$

Find the state of strain at the point P(0, 0.5) within the element. [40%]

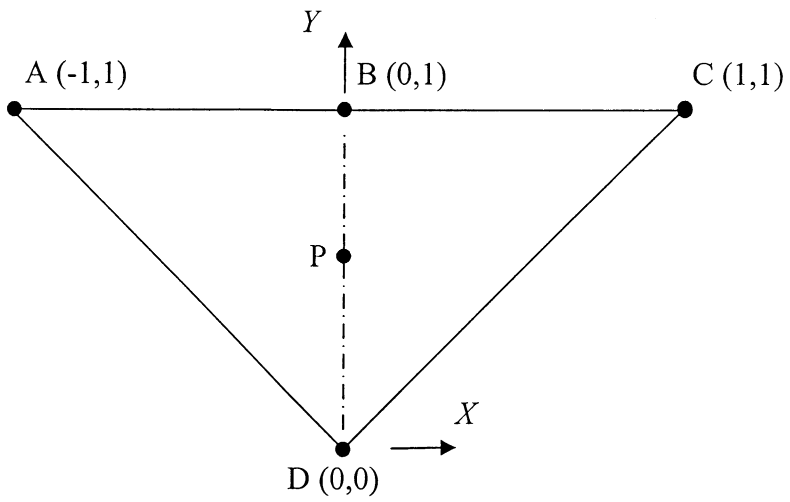

Fig. 3

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4 (a) The governing equation of one dimensional steady-state heat transfer can be expressed as follows:

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right)+Q=0$$

where T is the temperature, A is the cross-sectional area, k is the thermal conductivity and Q is the heat source or sink. The heat flux q at the end boundaries is $q = q_a$ at $x = a$ and $q = q_b$ at $x = b$. Show that the weak form of the governing equation is

$$\int_b^a \frac{dv}{dx}Ak\frac{dT}{dx}dx = (vA)_{x=b}q_b - (vA)_{x=a}q_a + \int_b^a vQdx$$

where v is a weight function. [30%]

(b) The temperature T and the weight function v are approximated using the following shape functions.

$$T = \mathbf{Na}, \quad \frac{dT}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{a} = \mathbf{Ba}$$

$$v = \mathbf{Nc}, \quad \frac{dv}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{c} = \mathbf{Bc}$$

where **N** is the shape function matrices, **a** is the nodal temperature values in vector form and **c** is the arbitrary nodal values in vector form. Show that the finite element approximation of the weak form given in part (a) becomes

$$\left(\int_b^a \mathbf{B}^{\mathrm{T}} Ak\mathbf{B}dx\right)\mathbf{a} = (\mathbf{N}^{\mathrm{T}}A)_{x=b}q_b - (\mathbf{N}^{\mathrm{T}}A)_{x=a}q_a + \int_b^a \mathbf{N}^{\mathrm{T}}Qdx$$

[20%]

(c) Four radioactive wastes are contained in a cylindrical carbon steel container with cement filler as shown in Fig. 4. The container is placed inside a circular tunnel excavated in a deep rock formation and backfilled with clay. The wastes continue to produce heat, and this is expected to raise the container's temperature. Finite element analysis is performed to investigate the steady-state temperature distribution inside the waste containment system.

(cont.

The temperature of the wastes is T_s, whereas the temperature of the rock far away from the waste containment system is T_r. Considering that the system is infinitely long, sketch a finite element mesh that can be used to compute the steady state temperature distribution inside the system. The outer boundary of the mesh should be taken at the interface of clay barrier and rock. In order to reduce computational time, consider symmetry as much as possible. [20%]

(d) List any assumptions needed to achieve the condition given in the finite element model developed in part (c). [15%]

(e) Define the boundary conditions of the finite element model developed in part (c). [15%]

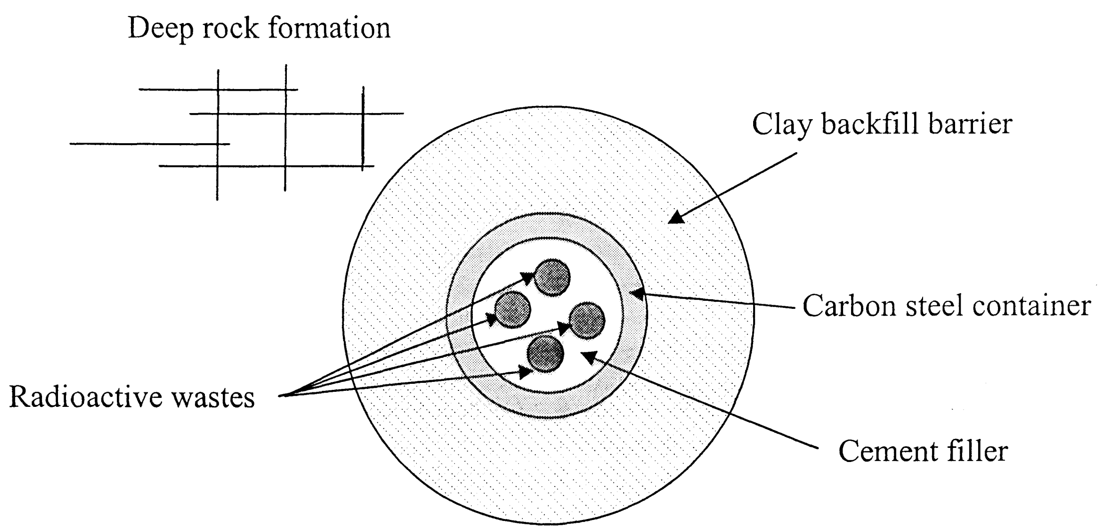

Fig. 4

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