ENGINEERING TRIPOS PART IIA

Wednesday 4 May 2005 2.30 to 4

Module 3E4

MODELLING CHOICE

*Answer not more than **two** questions.*

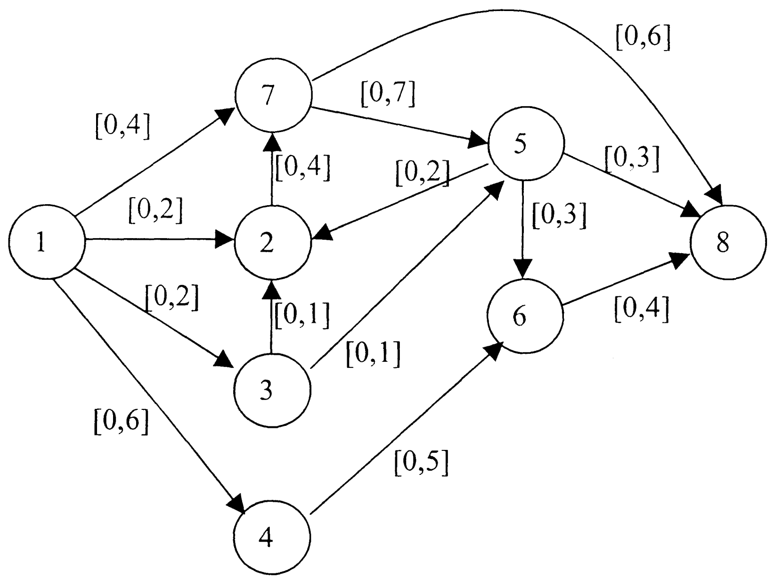
All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Consider the following network:



where for each arc (i,j) the attached label $[f_{ij},c_{ij}]$ contains the value of the current flow along the arc and the arc capacity, respectively.

(i) Determine the maximum flow that is possible to send between node 1 and node 8, by means of the Ford-Fulkerson algorithm. You are expected to use the current flow as a starting feasible solution, and to carry out the labeling algorithm for the first iteration in order to find an augmenting path from node 1 to node 8. Any other augmenting path, if needed, may be identified by simply inspecting the network. [45%]

(ii) Verify the optimality of the solution found by exploiting the "Min Cut – Max Flow" theorem. [25%]

(cont.

(b) Given the following multi-objective linear program:

$$
\begin{aligned}
\text{Max} \quad & 3x_1 + 2x_2 \\
\text{Max} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & x_1 - x_2 \leq 7 \\
& x_1 + 2x_2 \leq 16 \\
& 0 \leq x_1 \leq 9 \\
& 0 \leq x_2 \leq 6
\end{aligned}
$$

(i) Draw both the feasible region and the objective space, and give the coordinates of all vertices. [15%]

(ii) Determine the Pareto optimal frontier of the feasible region. [15%]

(TURN OVER

2 (a) A factory produces two types of tyres, "Rain" and "Slick", for the car manufacturing market, by applying a quarterly-based production management schedule. Over the next three months, May, June, and July, the factory needs to satisfy the following expected requests of each tyre (expressed in number of tyres requested per month):

Month	Rain Tyre	Slick Tyre
May	4,500	3,000
June	5,000	3,500
July	2,000	8,000

Two alternative machines, M1 and M2, are available in order to produce each tyre. Time availability (expressed in hours) of each machine for each month is reported in the table below:

Month	M1	M2
May	500	600
June	650	700
July	300	450

Moreover, production times vary depending on both the tyre type and the machine adopted. Such times are reported in the following table (expressed in hours):

Tyre type	M1	M2
Rain	0.10	0.12
Slick	0.18	0.15

The production cost per working hour is the same for both the machines, equal to £12 per hour. The raw material costs per each tyre are £4 and £5, respectively, for "Rain" and "Slick". The factory is trying to optimise the product mix over the next three months, particularly taking into account the possibility of producing more than the expected requests in the first two months, namely any over-production can be stored to be used in a subsequent month. The unit cost for carrying stock is 50p per month. Assuming that stored products are not available at the beginning of May, define decision variables and formulate this problem as a cost-minimising linear program to plan the production over the next three months, neglecting integrality of products. You are not required to solve this problem. [40%]

(cont.

(b) Consider the following linear program:

$$\begin{aligned} \text{Max} \quad & -2x_1 - x_2 - 3x_3 + 2x_4 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 4x_4 - x_5 = 1 \\ & x_1 + 2x_3 + 2x_4 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

(i) Explain why the point (1,0,0,0,2) is a basic feasible solution, and show that it is not optimal. [10%]

(ii) Apply the Simplex method to find the optimal solution of the problem. Use (1,0,0,0,2) as starting point. [25%]

(iii) Calculate the shadow prices for the two equality constraints. Explain the meaning of a shadow price. [10%]

(c) Let x^* and $x^\#$ be two (different) optimal solutions of a linear program. Show that in such a case the linear programming problem has infinitely many solutions. [15%]

(TURN OVER

3 (a) Consider the following integer linear program (ILP).

$$\begin{array}{ll} \text{Min} & z = 4x + y \\ \text{subject to} & 3x + y \geq 3 \\ & 4x + 3y \geq 6 \\ & x + 2y \leq 4 \\ & x, y \geq 0, \text{ integer} \end{array}$$

(i) Find the optimal solution of the ILP by using the branch-and bound algorithm. You may use the graphical approach for solving linear programs. Whenever you have the choice, you must branch first on the x-variable. Moreover, you are expected to use the breadth-first strategy to explore the branch-and-bound tree, and to specify the order of nodes that you visited in the whole branch-and-bound process. (Carry out the B&B method until you have either investigated 10 nodes or solved the problem). [45%]

(ii) Explain why the rounding heuristics may give bad results when used to solve integer linear programs. In particular apply the rounding technique to the optimal solution of the RLP, namely of the problem obtained from ILP by neglecting the integrality constraint, rounding up, down or to the nearest integer point. [10%]

(b) A distribution company has to supply 5 customers, each located at a different place in England. In order to improve supply services the company wants to choose at most 2 out of 3 available sites where to locate its deposits. Each deposit has a fixed construction cost, dependent on the selected site, and a maximum capacity, both listed in the following table:

Deposit	**Construction cost**	**Capacity**
D1	£10,000	180 tons
D2	£15,000	230 tons
D3	£13,000	500 tons

Each customer is expected to issue the following requests:

	Customer 1	**Customer 2**	**Customer 3**	**Customer 4**	**Customer 5**
Requests	91 tons	170 tons	135 tons	153 tons	110 tons

Moreover the unit transportation costs from each deposit to each customer are listed in the following:

(cont.

	Customer 1	Customer 2	Customer 3	Customer 4	Customer 5
D1	£15	£13	£27	£9	£7
D2	£12	£21	£34	£21	£3
D3	£7	£10	£2	£17	£12

Assuming there are no upper bounds on the quantities to be transported, formulate a linear model aimed at minimising the total operating cost (construction costs + transportation costs) while satisfying the requests issued by the customers. [45%]

(TURN OVER

4 (a) Consider the following nonlinear optimisation problem

$$
\begin{aligned}
\text{Min} \quad & x_2 \\
\text{subject to} \quad & (x_1 - 2)^2 + x_2^2 \le 4 \\
& x_1 - 2x_2 \ge 0 \\
& x_1 + x_2 \le 3 \\
& x_1 + x_2 \ge 0
\end{aligned}
$$

(i) Draw the feasible region and the level curves of the objective function and then solve the problem graphically. [25%]

(ii) Is the problem convex? Explain. [10%]

(iii) Are the constraints regular at the minimum point? [15%]

(iv) Write down the Karush-Kuhn-Tucker (KKT) conditions and verify that the minimum point satisfies these conditions. [20%]

(b) Consider the following linear program

$$
\begin{aligned}
\text{Min} \quad & c^{\mathrm{T}}x \\
\text{subject to} \quad & Ax = b \\
& x \ge 0
\end{aligned}
$$

where $x \in \Re^n$, $c \in \Re^n$, $A \in \Re^{m \times n}$, and $b \in \Re^m$. Denote by $\lambda \in \Re^m$ and $s \in \Re^n$ the multiplier vectors related to the equality and inequality constraints and write down the KKT conditions for this problem. Then, let (x^*, λ^*, s^*) be a KKT point of such a program and show that $(\lambda^*)^{\mathrm{T}} b = c^{\mathrm{T}} x^*$. Use this result to show that x^* is a global solution of the linear program. [30%]

END OF PAPER

PART IIA 2005

3E4 Modelling choice

Principal Assessor: **Dr G. Giallombardo JIMS**

Engineering Tripos Part IIA – 2005

Module 3E4 – Modelling Choice

Exam Paper

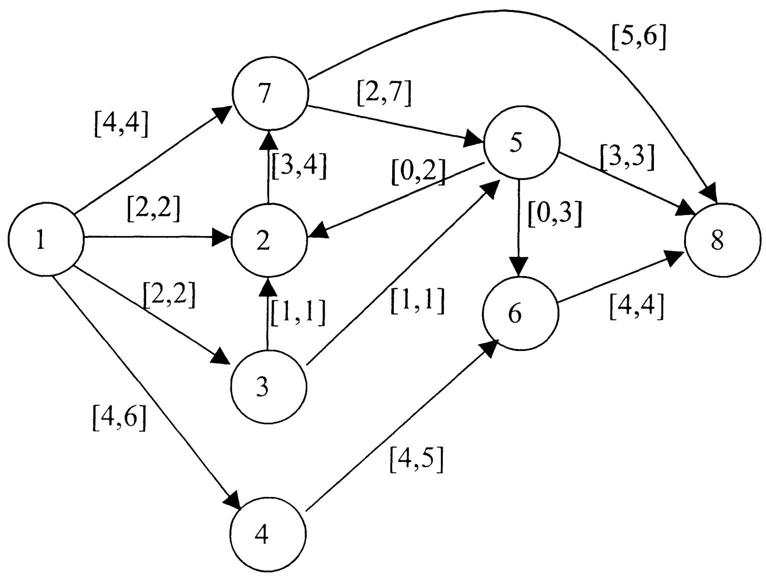
List of numerical answers

Principal Assessor: Dr G Giallombardo

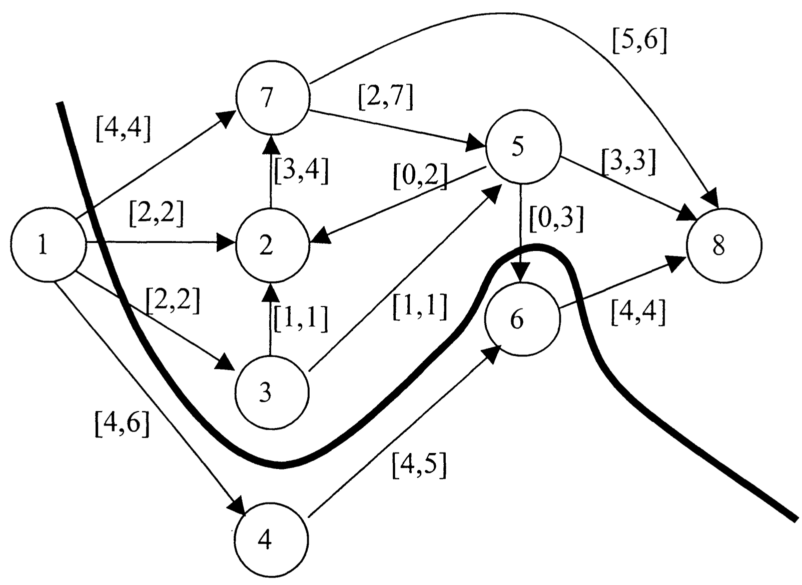
1.

(a)

(i) v=12



(ii) U={1,4,6} , W={2,3,5,7,8}



(b)

(i) $x^{(0)}=(0,0)$, $x^{(1)}=(0,6)$, $x^{(2)}=(4,6)$, $x^{(3)}=(9,7/2)$, $x^{(4)}=(9,2)$, $x^{(5)}=(7,0)$,
$z^{(0)}=(0,0)$, $z^{(1)}=(12,30)$, $z^{(2)}=(24,38)$, $z^{(3)}=(34,71/2)$, $z^{(4)}=(31,28)$, $z^{(5)}=(21,14)$.

(ii) Pareto solutions are all the points lying along the segment joining $x^{(2)}$ and $x^{(3)}$

2.

(a) Minimize
$1.2*(x_{11}+x_{12}+x_{13}) + 1,44*(x_{21}+x_{22}+x_{23}) + 2.16*(y_{11}+y_{12}+y_{13}) +1,8*(y_{21}+y_{22}+y_{23}) +$
$4*(x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23}) + 5*(y_{11}+y_{12}+y_{13}+y_{21}+y_{22}+y_{23}) +$

$0.50^*(x_1 + x_2 + y_1 + y_2)$

Subject to:

$0.10^* x_{11} + 0.18^* y_{11} \leq 500$
$0.10^* x_{12} + 0.18^* y_{12} \leq 650$
$0.10^* x_{13} + 0.18^* y_{13} \leq 300$

$0.12^* x_{21} + 0.15^* y_{21} \leq 600$
$0.12^* x_{22} + 0.15^* y_{22} \leq 700$
$0.12^* x_{23} + 0.15^* y_{23} \leq 450$

$x_{11} + x_{21} = 4500 + x_1$
$x_{12} + x_{22} + x_1 = 5000 + x_2$
$x_{13} + x_{23} + x_2 = 2000$

$y_{11} + y_{21} = 3000 + y_1$
$y_{12} + y_{22} + y_1 = 3500 + y_2$
$y_{13} + y_{23} + y_2 = 8000$

$x_{ij} \geq 0$, $i=1,2; j=1,2,3;$
$y_{ij} \geq 0$, $i=1,2; j=1,2,3;$
$x_i \geq 0$, $i=1,2;$
$y_i \geq 0$, $i=1,2.$

(b)

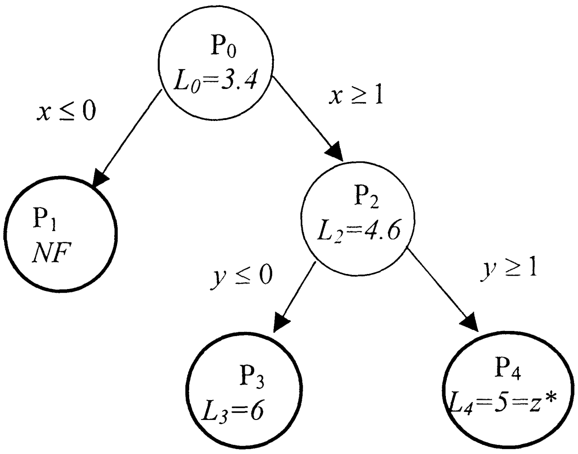
(i) Reduced cost vector $r^T = (p_N)^T - (p_B)^T B^{-1} N = (0, -2, 5)$.

(ii) Optimal solution (0,0,0,2/3,5/3).

(iii) $\lambda = (1/3,1/3)$.

3.

(a) Solution at (1,1) with value $L_4 = 5$.



(ii) Rounding up, down and to the nearest yields, respectively, (1,2), (0,1) and (0,2).

(b)

Minimize

$15x_{11} + 13x_{12} + 27x_{13} + 9x_{14} + 7x_{15} + 12x_{21} + 21x_{22} + 34x_{23} + 21x_{24} + 3x_{25} + 7x_{31} + 10x_{32} + 2x_{33} + 17x_{34} + 12x_{35} + 10{,}000y_1 + 15{,}000y_2 + 13{,}000y_3$

Subject to:

$x_{11} + x_{21} + x_{31} = 91$

$x_{12} + x_{22} + x_{32} = 170$

$x_{13} + x_{23} + x_{33} = 135$

$x_{14} + x_{24} + x_{34} = 153$

$x_{15} + x_{25} + x_{35} = 110$

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \le 180\, y_1$

$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \le 230\, y_2$

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \le 500\, y_3$

$y_1 + y_2 + y_3 \le 2$

$x_{ij} \ge 0,\ i=1,2,3; j=1,2,3,4,5;$

$y_i \in \{0,1\},\ i=1,2,3.$

4.

(a) Optimal solution at (2,–2).