ENGINEERING TRIPOS PART IIA

Tuesday 3 May 2005 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 Consider a causal discrete-time, linear system with a system transfer function

$$H(z) = \frac{z^2}{(z-p)(z-\bar{p})} \qquad \text{where } p = 0.9\, e^{j\pi/4} \text{ and } \bar{p} = 0.9\, e^{-j\pi/4}$$

In the questions that follow, the term *approximately* implies that you should not have to use a calculator, and that you should lucidly explain all approximations.

(a) Draw a pole-zero diagram corresponding to $H(z)$. Is the system stable? [20%]

(b) Show that θ_{max}, the value of θ, where $z = e^{j\theta}$, which maximises the magnitude response, is approximately $\pi/4$. [20%]

(c) Approximately determine A_{max} and Φ_{max}, the values of the magnitude and the phase angle of the frequency response at θ_{max}. [30%]

(d) Suppose the input to the filter is $u_k = 2\cos(\theta_{max}\, k)$, $k \geq 0$. Write down an approximate expression for the steady-state output sequence y_k. [10%]

(e) Approximately determine the frequency range over which the magnitude response is within a factor $1/\sqrt{2}$ of its maximum value. [20%]

2 (a) The block diagram of a discrete-time feedback control system is shown in Fig. 1.

(i) Determine the range of values for K_1 such that the closed-loop system is stable. [15%]

(ii) For the range of values of K_1 found in (i), find $\lim_{k\to\infty} e_k$ when the input $\{u_k\}$ is a unit step. [25%]

(iii) What values of K_1 result in a steady-state error of less than 1% ? [10%]

(b) A random process X is defined by

$$X(t,\alpha) = U(\alpha)\cos(\omega_0 t) + V(\alpha)\sin(\omega_0 t)$$

where $U(\alpha)$ and $V(\alpha)$ are independent random processes with zero mean and standard deviations σ_U and σ_V respectively, and ω_0 is a constant.

(i) Define the meaning of the term *wide sense stationary* (WSS) when applied to random processes that are functions of time. [10%]

(ii) Show that X is WSS if $\sigma_U = \sigma_V$. [40%]

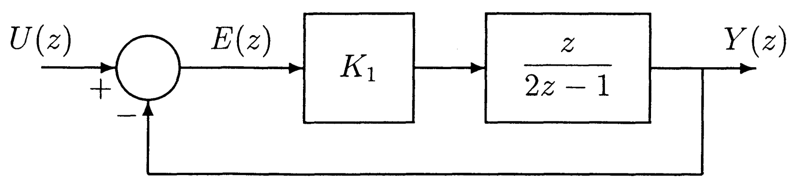

Fig. 1

3 (a) Explain the meaning of *ergodic* when applied to a stationary random process. [10%]

(b) A zero-mean ergodic random signal $X(t)$ has an autocorrelation function given by

$$r_{XX}(\tau) = \rho\,\delta(\tau)$$

where ρ is a constant. This signal is passed through a linear system with impulse response $h(t)$. Obtain an expression for $r_{YY}(\tau)$, the autocorrelation function of the system's output signal $Y(t)$, in terms of h and ρ . [30%]

(c) Hence calculate $r_{YY}(\tau)$ for the case when

$$h(t) = \begin{cases} \frac{1}{T}\exp(-t/T) & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

[30%]

(d) Describe how the power spectral density (PSD) of Y relates to $r_{YY}(\tau)$, and obtain an expression for this PSD. [30%]

4 (a) What is meant by the *mutual information* between two random variables? [15%]

(b) A memoryless source S produces a stream of symbols $S_1, S_2, S_3, \ldots$ with each symbol S_i taking the following values with the following probabilities:

S_i	P
A	0.7
B	0.2
C	0.07
D	0.03

A second source X emits symbols X_i each of which takes the value 0 or 1 with probabilities that depend on the value produced by S_i, given by the following table:

S_i	P(X_i=0)	P(X_i=1)
A	1.0	0.0
B	0.5	0.5
C	0.0	1.0
D	0.0	1.0

Calculate the entropies of S_i and X_i and the mutual information between X_i and S_i. [35%]

(c) Produce a Huffman code for S and find the efficiency of this code. If the source were extended to order four and a Huffman code created, how would you expect the efficiency of this new code to compare to your code? [30%]

(d) Now suppose that it is discovered that these sources are not memoryless and that the even numbered symbols emitted by X are always the same as the preceeding symbol (i.e., $X_1 = X_2$, $X_3 = X_4$, etc). Assuming the statistics given in the table above are unchanged, calculate the entropy of (S_1, S_2, X_1, X_2). [20%]

END OF PAPER

Module 3F1, April 2005 – SIGNALS AND SYSTEMS – Answers

1 (a) Stable.

(c) $A_{max} \approx \frac{10}{\sqrt{1.81}}$

$\Phi_{max} \approx -\frac{\pi}{4}$

(d) $y_k \approx \dfrac{20}{\sqrt{1.81}} \cos(\frac{\pi}{4}k - \frac{\pi}{4})$

(e) Bandwidth approximately covers the range $\pi/4 \pm 0.1$

2 (a) (i) $K_1 < -3$ or $K_1 > -1$

(ii) $\lim_{k\to\infty} e_k = \dfrac{1}{K_1 + 1}$

(iii) $K_1 < -101$ or $K_1 > 99$

3 (b) $r_{YY}(\tau) = \rho \int_{-\infty}^{\infty} h(\beta_1)\, h(\tau + \beta_1)\, d\beta_1 = \rho\, h(\tau) * h(-\tau)$

(c) $r_{YY}(\tau) = \dfrac{\rho}{2T} \exp\left(\dfrac{-|\tau|}{T}\right)$

(d) $\mathcal{S}_Y(\omega) = \dfrac{\rho}{1 + \omega^2 T^2}$

4 (b) $H(S_i) = 1.245$, $H(X_i) = 0.722$, $I(X_i; S_i) = 0.522$

(c) $\eta = 0.8893$, and tends towards unity if order of source increases.

(d) $H(S_1, S_2, X_1, X_2) = 2.168$

PART IIA 2005

3F1 Signals and systems

Principal Assessor: **Dr N G Kingsbury**