UNIVERSITY OF CAMBRIDGE

ENGINEERING TRIPOS PART IIA

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Wednesday 4 May 2005 9 to 10.30

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Module 3F2

SYSTEMS AND CONTROL

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 (a) Define the *matrix exponential function* $e^M$ , where $M$ is a square matrix. [10%]

(b) If $A$ is a square matrix and $t$ is a scalar, verify that $\frac{d}{dt}\{e^{At}\} = Ae^{At}$ . [10%]

Hence verify that $x(t) = e^{At}x(0)$ is a solution of the differential equation $\dot{x} = Ax$ , if $x$ is a vector. [10%]

(c) A 'double integrator' satisfies the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

where $x_1$ , $x_2$ and $u$ are scalars. Find the matrices $A$ and $B$ if these equations are written in the form

$$\dot{x} = Ax + Bu$$

where $x$ is the state vector $x = [x_1, x_2]^T$ . [5%]

Evaluate $e^{At}$ . [25%]

(d) A digital-to-analog converter produces the input $u(t)$ to the double integrator of part (c), and updates it at intervals of $T$ seconds, so that

$$u(t) = u_k \quad \text{for} \quad kT \leq t < (k+1)T, \quad k = 0, 1, 2, \ldots$$

If $x_k$ denotes the state vector $x$ at time $kT$ , $x_k = x(kT)$ , show that the matrices $F$ and $G$ , such that

$$x_{k+1} = Fx_k + Gu_k$$

are given by [40%]

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} .$$

2 (a) A linear system is given in state-space form as

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{1}$$

Explain what is meant by an *observer* for this system. [20%]

(b) Show that the convergence to zero of the state estimation error of an observer depends on the eigenvalues of the matrix $(A - LC)$, where $L$ is the observer gain matrix. [20%]

(c) The heading $\psi$ of an aircraft is measured by a compass, but with a lag of time-constant $T$, so that the measurement $\psi_c$ is determined by the equation

$$T\dot{\psi}_c + \psi_c = \psi.$$

The heading rate $\dot{\psi}$ is measured by a rate gyro instantaneously, but with an unknown constant bias $b$, so that the measurement $\dot{\psi}_g$ is given by

$$\dot{\psi}_g = \dot{\psi} + b.$$

This rate gyro measurement is integrated (exactly) to give a measurement $\psi_g$ of $\psi$, and the measurement error $\delta$ is defined as

$$\delta = \psi_g - \psi.$$

Taking $x = [\delta, b, \psi_c]^T$, $u = \psi$, and $y = [\psi_g, \psi_c]^T$, show that if the system equations are written in the form (1) then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{T} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and find the matrices $B$ and $D$. [30%]

(d) Consider an observer for the system in part (c). Show that any set of observer eigenvalues can be achieved if the observer gain matrix has the form [15%]

$$L = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & 0 \\ 0 & \ell_{32} \end{bmatrix}.$$

What would be the benefits of having an observer for this system? [15%]

3 A bicycle frame has a heading angle $\psi$ with respect to a fixed frame of reference, a roll angle $\phi$ from the vertical, and the steering angle is $\delta$, as shown in Fig.1. If the forward speed is $v$ and assumed to be constant, the equations of motion, for small $\psi$, $\phi$ and $\delta$, are given approximately by:

$$\dot{\psi} = \frac{v}{p}\delta$$
$$\ddot{\phi} = \frac{g}{h}\phi + \frac{v^2}{ph}\delta + \frac{bv}{ph}\dot{\delta}$$

where $b$, $h$ and $p$ are positive constants related to the geometry of the bicycle, and $g$ is the acceleration due to gravity.

(a) Introducing the variables $u = \dot{\delta}$ and $r = \dot{\phi}$, and defining the state vector

$$x = [\delta, \psi, \phi, r]^T$$

write the equations of motion in standard state-space form, with $u$ as the input. [20%]

(b) Analyse the open-loop stability of the bicycle. [25%]

(c) Show that the bicycle is controllable from $u$, providing that $v \neq 0$ and $v \neq b\sqrt{g/h}$. [25%]

(d) It is conjectured that a rider controls the bicycle (when following a straight path) by sensing only the heading $\psi$ and the roll rate $r$. Determine whether this is possible. [30%]

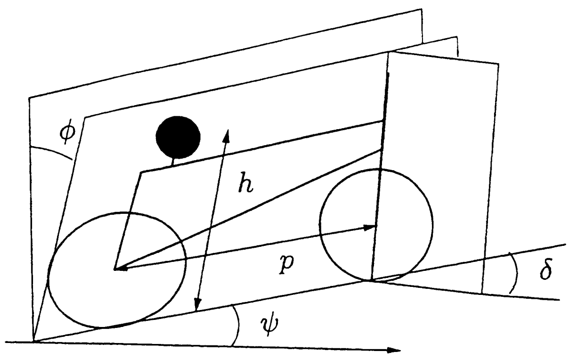


Fig. 1

4 Figure 2 shows a simple crane. A torque $T$ is exerted at the bottom of the crane, which results in the load moving laterally, with position $y$. The crane makes an angle $\theta$ with the vertical, and the cable carrying the load makes an angle $\psi$ with the vertical, as shown. When the crane is operating near the operating point $(\theta = \theta_0, \psi = 0, T = T_0)$ the linearised equations lead to the following transfer functions from $T$ to $\theta$ and from $\theta$ to $y$:

$$\frac{\bar{\theta}(s)}{\bar{T}(s)} = \frac{a}{s^2 - \omega_1^2} \qquad \text{and} \qquad \frac{\bar{y}(s)}{\bar{\theta}(s)} = \frac{b}{s^2 + \omega_2^2}$$

where $a$ and $b$ are appropriate positive constants.

(a) It is proposed to use proportional feedback from $y$ to $T$ to control the crane. Sketch the appropriate root-locus diagram if $\omega_1 = \omega_2$, and hence (or otherwise) show that this scheme will not work. [40%]

*Hint: The root-locus diagram consists entirely of straight-line segments in this case.*

(b) Consider the transfer function from $T$ to $\theta$ only. Show that if the angle $\theta$ and the angular velocity $\dot{\theta}$ are measured and fed back to the torque $T$ then it is possible to stabilise this subsystem. [30%]

(c) Figure 3 shows the root-locus diagram for the return ratio

$$L(s) = \frac{k(s + 0.5)}{(s + 6)^2(s^2 + 4)}$$

which is obtained with $\omega_2^2 = 4$, after the scheme of part (b) has been used to obtain two poles at $-6$, and the speed $\dot{y}$ has been measured and used to provide 'proportional and derivative' feedback. Estimate the range of gains $k$ for which closed-loop stability is obtained. [30%]

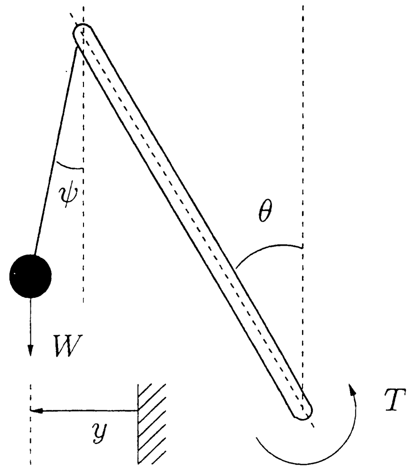


Fig. 2

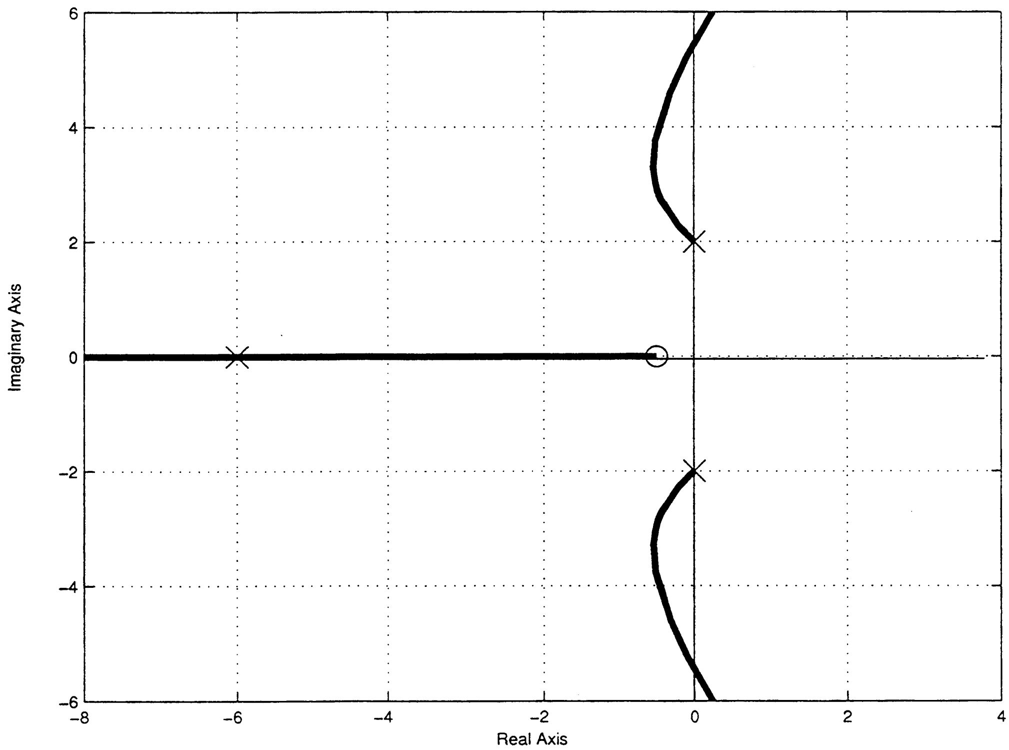


Fig. 3



**END OF PAPER**

PART IIA 2005

3F2 Systems and control

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# ENGINEERING TRIPOS PART IIA 2005

# ANSWERS TO MODULE 3F2: SYSTEMS AND CONTROL

1. (c): $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

2. (c): $B = \begin{bmatrix} 0 \\ 0 \\ 1/T \end{bmatrix}$, $\quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3. (a): $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ v/p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ v^2/ph & 0 & g/h & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ bv/ph \end{bmatrix}$$

   (b): Poles are at $0, 0, \pm\sqrt{g/h}$

   (d): Yes it is possible.

4. (c): $0 < k < 312$. (Since the question says 'estimate' and expects the use of graphical root-locus methods, any answer reasonably close to 312 is acceptable.)