ENGINEERING TRIPOS PART IIA

Saturday 14 May 2005 9 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version 2

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1 The Discrete Time Fourier Transform (DTFT) of a sequence $\{x_k\}$, $k = 0, \pm 1, \pm 2, \ldots, \pm\infty$ is given by

$$X(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-jk\omega}.$$

(a) Explain why the DTFT is not exactly computable on a digital computer. [20%]

(b) Consider now the Discrete Fourier Transform (DFT) of $\{x_k\}$ given for $n = 0, \ldots, N-1$ by

$$X_n = \sum_{k=0}^{N-1} x_k e^{-j\frac{2\pi kn}{N}}.$$

This DFT provides N spectrum values only, which is insufficient in some applications. Show how it is possible to compute

$$X'(\omega) = \sum_{k=0}^{N-1} x_k e^{-jk\omega}$$

as a function of $\{X_n\}$. In practice, we are often only interested in computing $X'(\omega)$ at the discrete frequencies $\frac{2\pi n}{P}$, where $n = 0, 1, \ldots, P-1$ and $P > N$. Name and describe an efficient way based on the DFT to compute these values. [40%]

(c) Assume you are interested in computing the DFTs of two real-valued sequences $\{x_k\}$ and $\{y_k\}$ of length N ($k = 0, 1, \ldots, N-1$). Show how it is possible to compute these DFTs using the DFT of a single complex-valued signal $\{z_k\}$ of length N given by

$$z_k = x_k + jy_k.$$

Which algorithm would be used to implement these DFTs if N is a power of 2? [40%]

2 (a) Describe the backward difference and bilinear transform methods to convert analog filters to digital filters. What are the advantages and disadvantages of the bilinear transform. [15%]

(b) For discrete time random processes $\{X_n\}$ and $\{Y_n\}$ define the terms: (i) autocorrelation function and cross-correlation function; (ii) wide sense stationarity. [15%]

(c) A highpass filter is required with sampling rate 32.20kHz and a 3dB corner frequency of 6.50kHz. We are interested in designing such an infinite impulse response (IIR) digital filter from an analogue prototype given by

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}.$$

Using the lowpass to highpass transformation

$$s \longrightarrow \frac{\Omega_c}{s},$$

where Ω_c is the corner frequency, together with the bilinear transform, design the required digital filter. [40%]

(d) A finite impulse response (FIR) filter has transfer function

$$H(z) = \left(1 - z^{-1}\right)^2.$$

A white noise process having variance equal to 1 is placed at the input to this filter. Determine the cross-correlation function between the white noise input process and the filtered output. Determine also the power of the output. [30%]

3 A signal x_n is observed in a noisy and reverberant environment

$$y_n = x_n + 0.4x_{n-1} + v_n,$$

where v_n is zero-mean white noise with power equal to 1 and x_n is an autoregressive process (AR) process, independent of v_n, having autocorrelation function

$$r_{XX}[k] = 0.9^{|k|}.$$

(a) It is desired to estimate x_n from measurements of y_n only by filtering with an infinite impulse response (IIR) filter having impulse response h_n, $n = -\infty, \ldots, +\infty$:

$$\hat{x}_n = \sum_{p=-\infty}^{+\infty} h_p y_{n-p}.$$

Show that the optimal Wiener filter coefficients must satisfy the equation:

$$\sum_{p=-\infty}^{+\infty} h_p r_{YY}[q-p] = r_{YX}[q], \quad -\infty < q < +\infty.$$

[40%]

(b) Determine $r_{YX}[p]$ and $r_{YY}[p]$. [30%]

(c) Hence determine the frequency response of the optimal Wiener filter in this case. [30%]

[In this question you may use the result:

$$\sum_{n=-\infty}^{+\infty} \alpha^{|n|} e^{-j\Omega n} = \frac{1-\alpha^2}{|1-\alpha\exp(-j\Omega)|^2}]$$

4 A classifier is to be trained for a two class problem. The priors for the two classes are equal and the data for class ω_1 is known to be Gaussian distributed with a mean of 0 and variance of 1. The class-conditional probability density function (PDF) for class ω_2 is to be trained using N independent 1-dimensional samples of training data from that class, $x_1, \ldots, x_N$. The form of the PDF is known to be a Gaussian distribution. The variance of the PDF for ω_2 is *assumed* to be equal to 1 and not re-estimated. Maximum likelihood (ML) training is to be used to estimate the mean of the distribution.

(a) Briefly discuss the issues that must be considered when estimating the parameters of a class-conditional PDF for use in a classification system. Under what conditions will the decision boundaries obtained using class-conditional PDFs estimated using ML training be the same as the Bayes' decision boundaries? [20%]

(b) Write down an expression for the log-likelihood of the N samples of training data for class ω_2 . Hence derive the ML estimate of the mean for the PDF of class ω_2 using the available training data. [25%]

(c) It is discovered that the data for class ω_2 is actually Gaussian distributed with a mean of 1 and a variance of 10^6 .

(i) What is the ML estimate of the mean that would be estimated in part (b) as the value of N gets large? [10%]

(ii) What equation is satisfied by a point x that lies on the Bayes decision boundary for this classification problem using the *correct* class-conditional PDFs for classes ω_1 and ω_2 ? [20%]

(iii) Using the result from section (c)(ii), obtain a new value for the mean of the estimated PDF for class ω_2 that will yield a lower error rate than the ML estimate in part (c)(i). Note the variance of the estimated PDF for class ω_2 is fixed at 1. [25%]

END OF PAPER