PART IIA 2006
3C5: Dynamics
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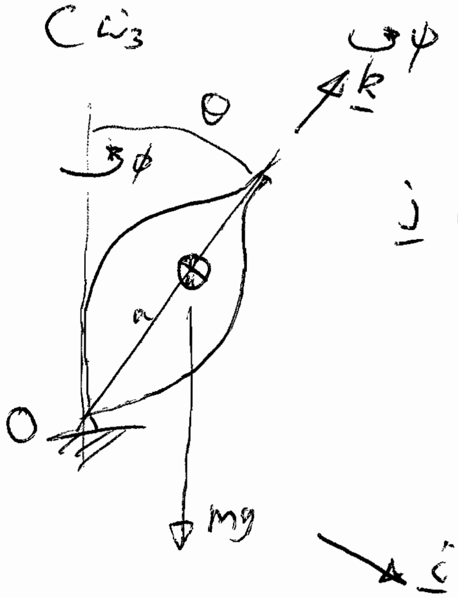
Datasheet: 3C5/3C6 Data Sheet: Dynamics & Vibration

Part IIA 3C5

1. Gyro equations :

$$A\dot{\omega}_1 - (A\Omega_3 - C\omega_3)\omega_2 = Q_1 \quad (1)$$
$$A\dot{\omega}_2 + (A\Omega_3 - C\omega_3)\omega_1 = Q_2 \quad (2)$$
$$C\dot{\omega}_3 = Q_3 \quad (3)$$



Use Euler angles $\theta\ \phi\ \psi$

Assume θ = const for steady precession

$\underline{j}$ into page

Moments about O

$Q_1 = 0$

$Q_2 = mga\sin\theta$

$Q_3 = 0$

(3) $\rightarrow$ ω_3 = constant = ω

Euler Angle relations : $\Omega_1 = \omega_1 = -\dot{\phi}\sin\theta$

$\Omega_2 = \omega_2 = \dot{\theta} = 0$ in S.S.

$\Omega_3 = \dot{\phi}\cos\theta$

$\omega_3 = \Omega_3 + \dot{\psi}$

<u>Steady state</u>

$\dot{\omega}_1 = 0 \qquad \dot{\omega}_2 = 0 \qquad , \omega_2 = 0$

$\therefore$ (1) ~~...~~ is $0 = 0$

(2) $\rightarrow$ $\cancel{-}(A\dot{\phi}\cos\theta - C\omega)\dot{\phi}\sin\theta = mga\sin\theta \quad (4)$

a/ Fast spin $\omega \gg \dot{\phi}$ $\quad\therefore\quad$ $\boxed{\dot{\phi} = \dfrac{mga}{C\omega}}$

(or $\sin\theta = 0$) (see later)

b/ (4) is a quadratic in $\dot{\phi}$

$$A\cos\theta\,\dot{\phi}^2 - C\omega\dot{\phi} + mga = 0$$

$\therefore$ solution if $(c\omega)^2 - 4A\cos\theta\, mga > 0$
(ie "$b^2 - 4ac$")

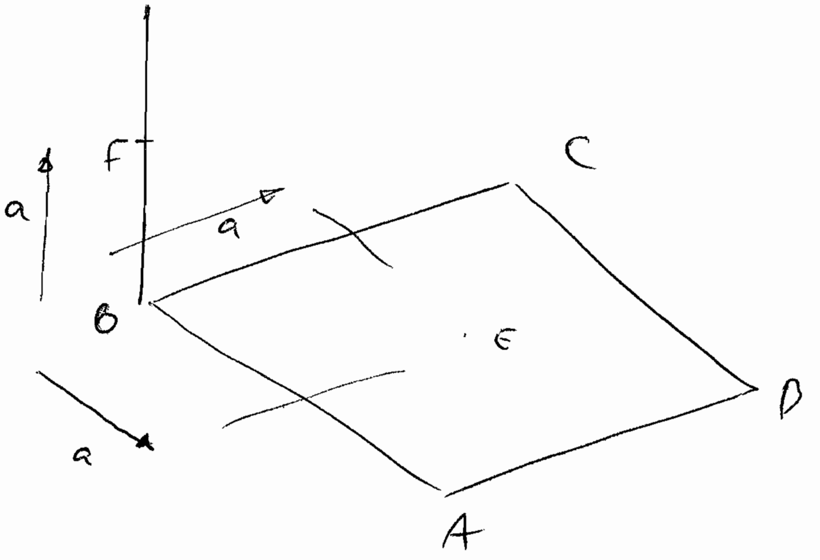
$$\therefore \quad \omega^2 > \frac{4\, mga A \cos\theta}{C^2}$$

Small θ $\rightarrow$ $\boxed{\omega^2 > \dfrac{4mgaA}{C^2}}$ $*$

Not valid for $\theta = 0$ since $\sin\theta$ was cancelled.
$\theta = 0$ is a solution for all θ
but is unstable for $\omega^2 <$ $*$.

9 minutes

2



(a) G is at mean of $(0,0,a)$ for bar (F)

and $(a,a,0)$ for plate (E)

$\therefore \quad \underline{r}_G = \frac{1}{2}(a,a,a)$

(b) I_{plate} at centre $= \begin{bmatrix} \frac{1}{3}6ma^2 & 0 & 0 \\ 0 & \frac{1}{3}6ma^2 & 0 \\ 0 & 0 & \frac{2}{3}6ma^2 \end{bmatrix}$

(from data book & perpendicular axis theorem)

$$= 2ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$I_O = I_G + m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

with $x = -a$, $y = -a$, $z = 0$

$$= 2ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + 6ma^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2ma^2 \begin{bmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

2....

For rod $I_A = \frac{4}{3} 6ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2ma^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore I_A(\text{total}) = 2ma^2 \begin{bmatrix} 8 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

(agrees with question: diag terms are $16ma^2$, products are either zero or $-6ma^2$)

(c) $I_A = I_G + 12M \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$

with $x = y = z = -\frac{a}{2}$

$\therefore I_G = 2ma^2 \begin{bmatrix} 8 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} - 12ma^2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}$

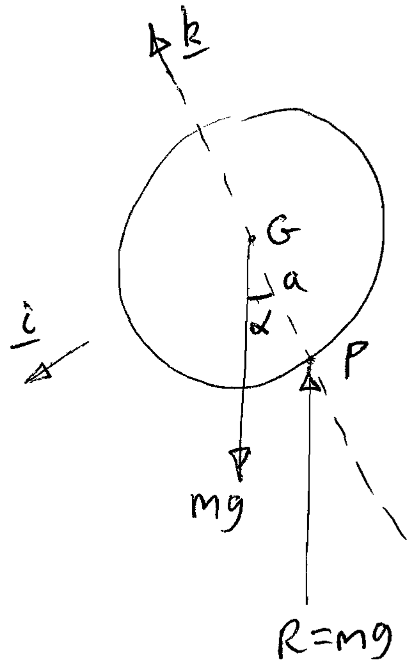
$= ma^2 \begin{bmatrix} 10 & -3 & 3 \\ -3 & 10 & 3 \\ 3 & 3 & 10 \end{bmatrix}$

(d) $\vec{EF} = (-1, -1, 1)^T$

Principal axis satisfies $[I_G]\,\vec{EF} = \lambda\,\vec{EF}$

$\therefore ma^2 \begin{bmatrix} -4 \\ -4 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \quad \therefore \underline{\lambda = 4ma^2}$ ✓

3(a)(i) The ball is moving at a constant speed so there is no net force. The only forces acting on the ball are $mg \downarrow$ and the reaction at $P \uparrow$



($\underline{j}$ out of page)

(ii) From the diagram the net couple on the ball $\underline{Q} = mga\sin\alpha\,\underline{j}$

$$\therefore Q_1 = Q_3 = 0$$

$$Q_2 = mga\sin\alpha$$

(iii) In Euler's equations using $\underline{i}\ \underline{j}\ \underline{k}$ as defined (fixed in space – possible because ball is "AAA") ie $A=B=C$

$$A\dot{\omega}_1 - \cancel{(B-C)}^{0}\omega_2\omega_3 = \cancel{Q_1}^{0}$$

$$B\dot{\omega}_2 - \cancel{(C-A)}^{0}\omega_3\omega_1 = Q_2$$

$$C\dot{\omega}_3 - \cancel{(A-B)}^{0}\omega_1\omega_2 = \cancel{Q_3}^{0}$$

$$\therefore \dot{\omega}_1 = 0, \quad \dot{\omega}_3 = 0$$

The only "interesting" equation is

$$A\dot{\omega}_2 = Q_2 \qquad \text{with } A = \tfrac{2}{5}ma^2 \text{ for sphere}$$

$$\therefore \dot{\omega}_2 = \frac{mga\sin\alpha}{\frac{2}{5}ma^2} \qquad \therefore \underline{\underline{\dot{\omega}_2 = \frac{5g\sin\alpha}{2a}}}$$

3(b) The ball is not accelerating at "G"

$$\dot{\underline{r}}_G = v\underline{j} = \text{constant}.$$

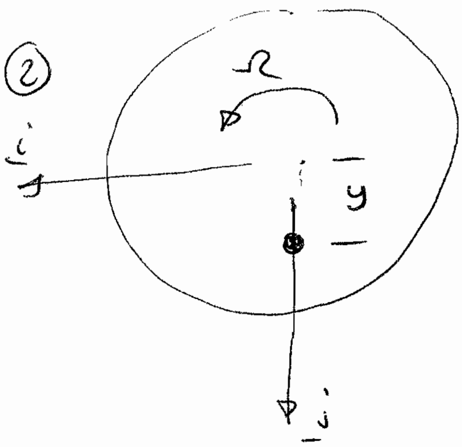
The velocity of P (contact point) is

$$\underline{v}_P = \dot{\underline{r}}_G + \underline{\omega} \times (-a\underline{k})$$

$$= v\underline{j} + a\omega_1\underline{j} - a\omega_2\underline{i} \qquad (1)$$

The velocity of the table along AB is

$$\underline{v}_T = -y\Omega\underline{i} \qquad (2)$$



No slip $\therefore \underline{v}_P = \underline{v}_T$

$$\therefore (v + a\omega_1)\underline{j} - a\omega_2\underline{i} = -y\Omega\underline{i}$$

$$\therefore \omega_1 = -\frac{v}{a} \quad \text{constant}$$

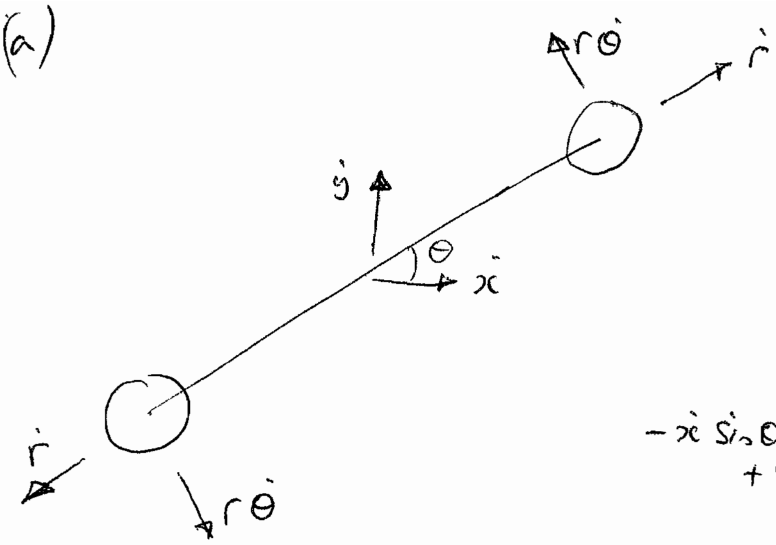
and $a\omega_2 = y\Omega$

but $\dot{\omega}_2$ is constant from (a)

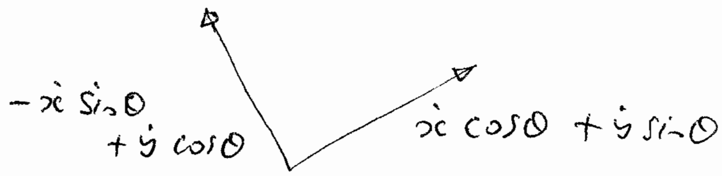
$$\therefore \dot{y} = \frac{a\dot{\omega}_2}{\Omega} = \frac{a\,5g\sin\alpha}{2a\Omega} = v$$

$$\therefore v = \frac{5g\sin\alpha}{2\Omega}$$

4(a)



resolve $\dot x$ & $\dot y$



$$T = \tfrac{1}{2}m\left[\left(\dot x\cos\theta + \dot y\sin\theta + \dot r\right)^2 + \left(-\dot x\sin\theta + \dot y\cos\theta + (r\dot\theta)\right)^2\right]$$

$$+ \tfrac{1}{2}m\left[\left(\dot x\cos\theta + \dot y\sin\theta - \dot r\right)^2 + \left(-\dot x\sin\theta + \dot y\cos\theta - (r\dot\theta)\right)^2\right]$$

$$= \tfrac{1}{2}m\left[\dot x^2 + \dot y^2 + \dot r^2 + (r\dot\theta)^2 + 2\dot r\left(\dot x\cos\theta + \dot y\sin\theta\right) + 2(r\dot\theta)^2\left(-\dot x\sin\theta + \dot y\cos\theta\right)\right]$$

$$+ \tfrac{1}{2}m\left[\dot x^2 + \dot y^2 + \dot r^2 + (r\dot\theta)^2 - 2\dot r\left(\dot x\cos\theta + \dot y\sin\theta\right) - 2(r\dot\theta)^2\left(-\dot x\sin\theta + \dot y\cos\theta\right)\right]$$

$$= m\left[\dot x^2 + \dot y^2 + \dot r^2 + (r\dot\theta)^2\right]$$

$$V = \frac{-GM^2}{2r}$$

$x:\quad \frac{\partial T}{\partial \dot x} = 2m\dot x \qquad \frac{\partial T}{\partial x} = 0 \qquad \frac{\partial V}{\partial x} = 0$ (1)

$y:\quad \frac{\partial T}{\partial \dot y} = 2m\dot y \qquad \frac{\partial T}{\partial y} = 0 \qquad \frac{\partial V}{\partial y} = 0$ (2)

$r:\quad \frac{\partial T}{\partial \dot r} = 2m\dot r \qquad \frac{\partial T}{\partial r} = 2mr\dot\theta^2 \qquad \frac{\partial V}{\partial r} = \frac{GM^2}{2r^2}$ (3)

$\theta:\quad \frac{\partial T}{\partial \dot\theta} = 2r^2\dot\theta \qquad \frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial V}{\partial \theta} = 0$ (4)

4(a) ① $\to$ $\dot{x} = \text{constant}$

② $\to$ $\dot{y} = \text{constant}$

③ $\to$ $2\ddot{r} - 2r\dot{\theta}^2 + \dfrac{GM}{2r^2} = 0$

④ $\to$ $r^2\dot{\theta} = \text{constat}$

4(b) ①, ② & ④ are the required results. These are conservation laws ① of momentum $\leftrightarrow$
② of momentum $\updownarrow$
③ of moment of momentum $\circlearrowright$

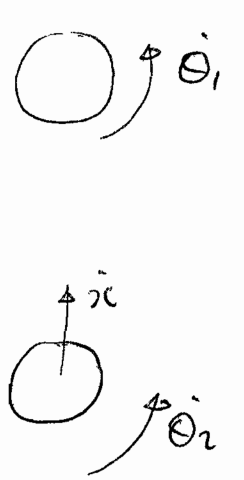
4(c) Steady motion with $r = \text{constant}$

$\therefore$ ③ $\to$ $\dot{\theta}^2 = \dfrac{GM}{4r^3}$

$$T = \frac{2\pi}{\dot{\theta}} = 2\pi\sqrt{\frac{4r^3}{GM}}$$

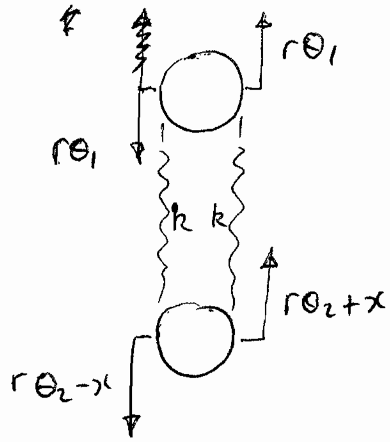
$$= \frac{4\pi r^{\frac{3}{2}}}{\sqrt{GM}}$$

5 (a)



$$T = \tfrac{1}{2} m r^2 \dot\theta_1^2 + \tfrac{1}{2} m \dot x^2 + \tfrac{1}{2} m r^2 \dot\theta_2^2$$

$$= \tfrac{1}{2} m\left(\dot x^2 + (r\dot\theta_1)^2 + (r\dot\theta_2)^2\right)$$



$$V = \tfrac{1}{2} k\left(\left(r\theta_2 - x - r\theta_1\right)^2 + \left(r\theta_1 - (r\theta_2 + x)\right)^2\right)$$

$$= \tfrac{1}{2} k\left(\left(x + r(\theta_1 - \theta_2)\right)^2 + \left(x - r(\theta_1 - \theta_2)\right)^2\right)$$

$$= \tfrac{1}{2} k\left(2x^2 + 2r^2(\theta_1 - \theta_2)^2\right)$$

$$\therefore \; [M] = \begin{bmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2 \end{bmatrix} \quad \& \quad [K] = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2kr^2 & -2kr^2 \\ 0 & -2kr^2 & 2kr^2 \end{bmatrix}$$

(b) $\underset{\sim}{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\omega = 0$ for rigid body rotation

$$[K]\underline{u} = [M]\omega^2 \underline{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{since } \omega = 0)$$

Check $[K]\underline{u} = 2k \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & -r^2 \\ 0 & -r^2 & r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

5(c) Put $\theta_1 = 0$ in T & V from part (a)

$$\therefore \quad [M] = \begin{bmatrix} m & 0 \\ 0 & mr^2 \end{bmatrix} \quad \& \quad [K] = \begin{bmatrix} 2k & 0 \\ 0 & 2kr^2 \end{bmatrix}$$

$$\left| [K] - [M]\omega^2 \right| = 0$$

$$\therefore \quad \begin{vmatrix} 2k - m\omega^2 & 0 \\ 0 & 2kr^2 - mr^2\omega^2 \end{vmatrix} = 0$$

$$\therefore \quad (2k - m\omega^2)^2 = 0$$

repeated eigenvalues $\omega = \sqrt{\frac{2k}{m}}$ for both modes

By symmetry, mode shapes are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ie pure bounce & pure rotation.