PART IIA 2006
3C6: Vibration
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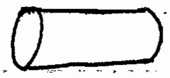
PART IIA PAPER

Datasheet: 3C5/3C6 Data Sheet Dynamics & Vibration

1.(a)

Torque $GJ\left.\dfrac{\partial\theta}{\partial x}\right|_x$ ↺  ↻ Torque $GJ\left.\dfrac{\partial\theta}{\partial x}\right|_{x+\delta x}$

x $\quad$ $x+\delta x$

where $J = \dfrac{\pi}{2} a^4$ (from data book)

So $$\underbrace{\rho J \,\delta x}_{\text{Polar moment of inertia of element}} \frac{\partial^2\theta}{\partial t^2} = GJ\left(\left.\frac{\partial\theta}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial\theta}{\partial x}\right|_{x}\right)$$

Divide by δx, let $\delta x \to 0$:

$$\rho J \frac{\partial^2\theta}{\partial t^2} = GJ\frac{\partial^2\theta}{\partial x^2} \quad \text{as data sheet.}$$

(b) Fixed at $x=0 \rightarrow \theta = 0$

Free at $x = L \rightarrow$ no torque $\rightarrow \dfrac{\partial\theta}{\partial x} = 0$

For a mode, let $\theta(x,t) = u(x)\,e^{i\omega t}$

Then $$-\rho\omega^2 u = G\frac{d^2u}{dx^2}$$

General solution $u = A\cos kx + B\sin kx$, $k^2 = \dfrac{\rho\omega^2}{G}$

$u = 0$ at $x=0 \rightarrow A = 0$

$u' = 0$ at $x = L \rightarrow \cos kL = 0$

$\therefore kL = (n+\tfrac{1}{2})\pi$, $n = 0, 1, 2 \cdots$

So modes are $u_n(x) = \sin\dfrac{(n+\tfrac12)\pi x}{L}$

with natural frequency $\omega_n = \sqrt{\dfrac{G}{\rho}}\,(n+\tfrac12)\dfrac{\pi}{L}$

(c) Rotor attached at $x = L$

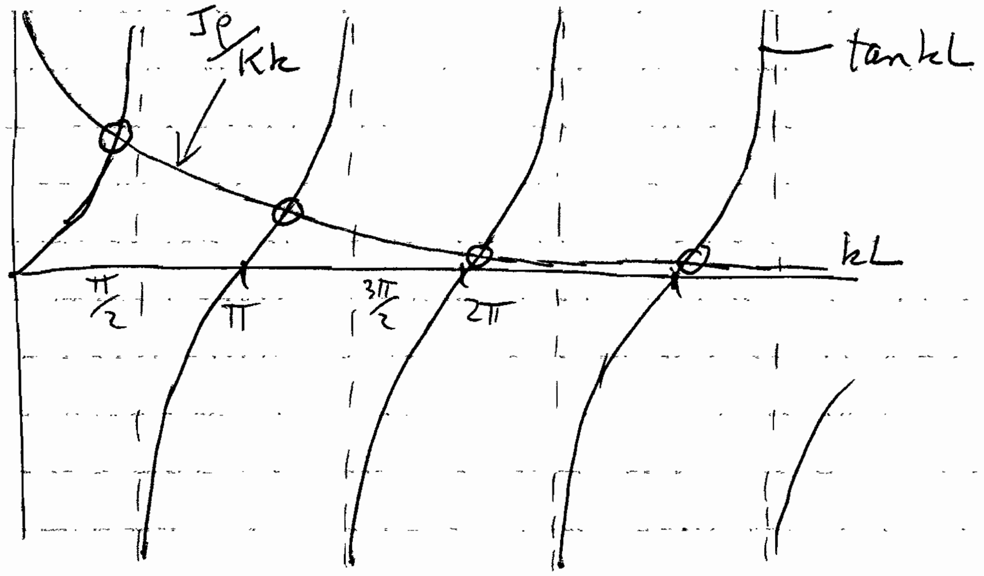
Apply Newton to rotor: $K\dfrac{\partial^2\theta}{\partial t^2} = -GJ\dfrac{\partial\theta}{\partial x}$ at $x = L$

1. cont. i.e. $-K\omega^2 u = -GJu'$ at $x = L$

so with $u = \sin kx$, $K\omega^2 \sin kL = GJ k \cos kL$

$$\therefore \tan kL = \frac{GJ}{K} \cdot \frac{\rho}{Gk} = \frac{J\rho}{Kk} \quad \text{with } \omega = k\sqrt{\frac{G}{\rho}}$$

(d)



Values of kL at natural frequencies are ringed. All are lower than corresponding values without the rotor, which are the values where $\tan kL \to \infty$. This lowering is expected since inertia has been added without changing stiffness.

The higher values tend towards the zeros of the tangent function. These correspond to $\sin kL = 0$, which gives the natural frequencies of a shaft fixed at both ends. The inertia of the rotor acts as a clamp at these high frequencies.

As $K \to \infty$ (heavy rotor) the lowest frequency $\to 0$. This is a slow mode with the rotor moving on the "spring" of a massless shaft.

2.(a) From data sheet, equation is

$$m\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f \qquad (1)$$

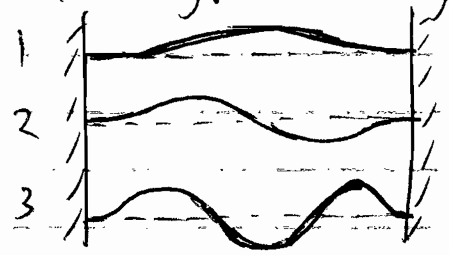
For modes, $f=0$ and $w = u(x)\,e^{i\omega t}$.

So require $\quad -m\omega^2 u + EI u'''' = 0$

Solutions are $\quad u = \left.\begin{matrix}\sin\\ \cos\end{matrix}\right)\alpha x\,,\ \left.\begin{matrix}\sinh\\ \cosh\end{matrix}\right)\alpha x\,,\ \alpha^4 = \dfrac{m\omega^2}{EI}$

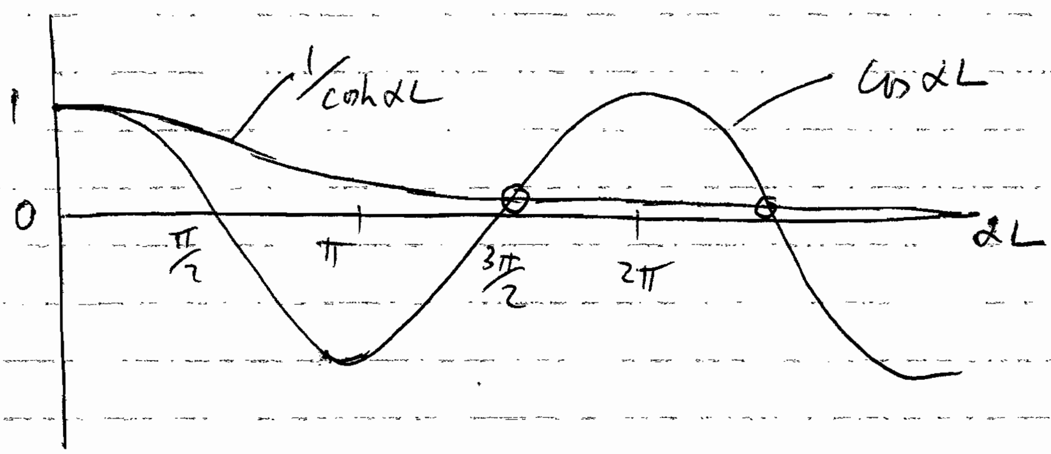
Take a general linear combination, with 4 constants. Use the boundary conditions $u=0, u'=0$ at $x=0, L$ to get 4 simultaneous equations for these constants. For non-trivial solution the determinant of this set must vanish. This gives the eigenvalue equation.

Modes:



(b) Given $\cos\alpha L \cosh\alpha L = 1$

$$\rightarrow \cos\alpha L = 1/\cosh\alpha L$$

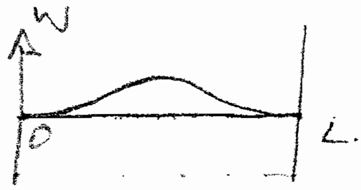


So $\alpha L \simeq (n+\tfrac{1}{2})\pi$, $n = 1,2,3\cdots$

So $\quad \omega_n \simeq \sqrt{\dfrac{EI}{m}}\,\dfrac{(n+\frac{1}{2})^2\pi^2}{L^2}$

2.(c) Put $w = x^2(L-x)^2$ into ①

$= x^2 - 2x^3L + x^4$



$\rightarrow f = EI\,24 = \text{constant}$

So this is the solution to static response to a uniform load, e.g self-weight.

Expression satisfies $w=0$, $w'=0$ at $x=0, L$ so it fits the clamped-clamped beam.

It "looks plausible" to approximate the lowest mode, so use Rayleigh.

From data sheet

$$V = \frac{1}{2} EI \int_0^L w''^2\,dx = \frac{1}{2} EI \int_0^L (12x^2 - 12xL + 2L^2)^2\,dx$$

$$= 2EI \int_0^L (36x^4 + 36x^2L^2 + L^4 - 72x^3L + 12x^2L^2 - 12xL^3)\,dx$$

$$= 2EI\left[36\frac{L^5}{5} + 36\frac{L^5}{3} + L^5 - 72\frac{L^5}{4} + 12\frac{L^5}{3} - 12\frac{L^5}{2}\right]$$

$$= 2EIL^5\left(\frac{36+60+5-90+20-30}{5}\right) = \frac{2}{5}EIL^5$$

$$\tilde{T} = \frac{1}{2} m \int_0^L w^2\,dx = \frac{1}{2} m \frac{L^9}{630} \quad \text{as given}$$

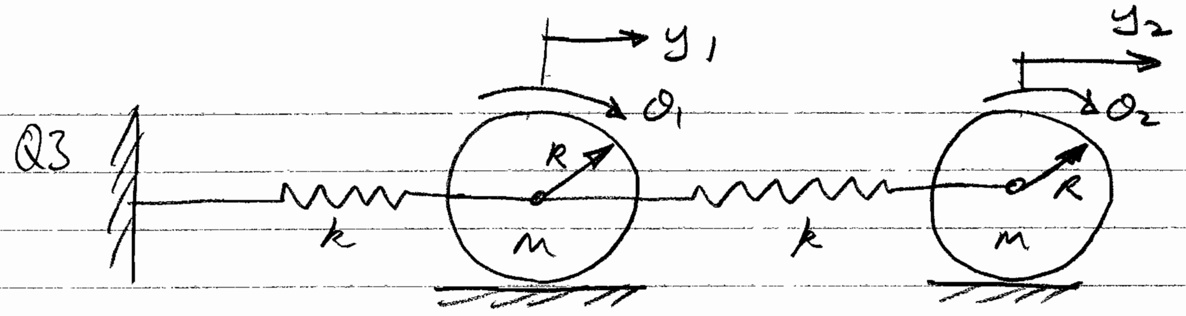
$$\text{So } \omega^2 \simeq \frac{V}{\tilde{T}} = \frac{EI}{mL^4} \cdot \frac{2}{5} \Big/ \left(\frac{1}{2} \times \frac{1}{630}\right) = 504 \frac{EI}{mL^4}$$

Approximate answer from (b) with $n=1$ is

$$\omega^2 \simeq \frac{EI}{mL^4}\left(\frac{3\pi}{2}\right)^4 \simeq 493 \frac{EI}{mL^4}$$

The two are very close, and we know from the graph in (b) that $493 \frac{EI}{mL^4}$ is a bit too low. On the other hand the Rayleigh estimate is too high, so true result must lie between the two.

Q3



(a) $T = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}J\dot{\theta}_1^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}J\dot{\theta}_2^2$

$$J = mR^2/2 \quad \& \quad \dot{y}_1 = R\dot{\theta}_1 \;,\; \dot{y}_2 = R\dot{\theta}_2$$

So $T = \frac{1}{2}m\dot{y}_1^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{\dot{y}_1}{R}\right)^2 + \frac{1}{2}m\dot{y}_2^2 + \frac{1}{2}\frac{mR^2}{2}\left(\frac{\dot{y}_2}{R}\right)^2$

$$T = \frac{1}{2}\left(\frac{3}{2}m\dot{y}_1^2 + \frac{3}{2}m\dot{y}_2^2\right) \Rightarrow [M] = \frac{3m}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$V = \frac{1}{2}ky_1^2 + \frac{1}{2}k(y_2 - y_1)^2$$
$$= \frac{1}{2}k(y_1^2 + y_2^2 - 2y_2y_1 + y_1^2)$$
$$= \frac{1}{2}k(2y_1^2 - 2y_1y_2 + y_2^2) \quad [K] = k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$$

(b) Eigenvalue problem is $\left([K] - \omega^2[m]\right)\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} = 0$

$$\begin{bmatrix}2k - \omega^2 3m/2 & -k\\ -k & k - \omega^2 3m/2\end{bmatrix}\begin{Bmatrix}y_1\\ y_2\end{Bmatrix} = 0$$

CE $\Rightarrow$ $(2k - \omega^2 3m/2)(k - \omega^2 3m/2) - k^2 = 0$

$$k^2 - \omega^2 3m/2\,(2k + k) + \omega^4\, 9m^2/4 = 0$$

ie $\frac{9m^2}{4}\omega^4 - \frac{9k}{2}m\omega^2 + k^2 = 0$

$$\omega^2 = \frac{\frac{\cancel{9}^{3}km}{2} \pm \sqrt{\frac{\cancel{81}^{9}}{4}k^2m^2 - 9m^2k^2}}{3\,\cancel{9}m^2/2}$$

$$\omega^2 = \frac{\frac{3km}{2} \pm \frac{km\sqrt{5}}{2}}{3m^2/2} = \frac{k}{m} \pm \frac{\sqrt{5}}{3}k/m = 0.255\,k/m,\ 1.745\,\frac{k}{m}$$

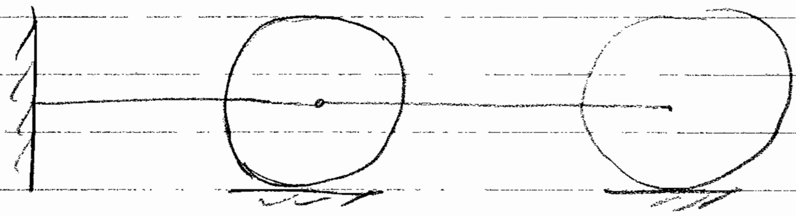
3. (Cont) Mode shapes

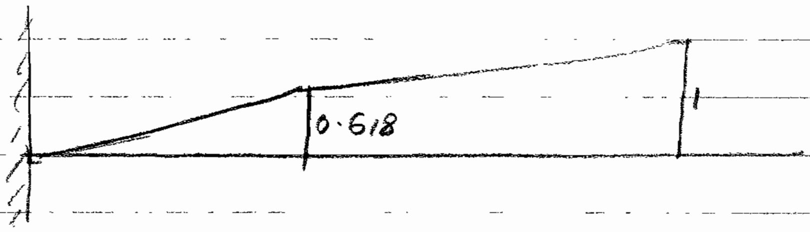
$$\left(2k - \omega^2 \frac{3m}{2}\right) y_1 - k y_2 = 0$$

$$\Rightarrow \frac{y_1}{y_2} = \frac{k}{2k - \omega^2 3m/2}$$

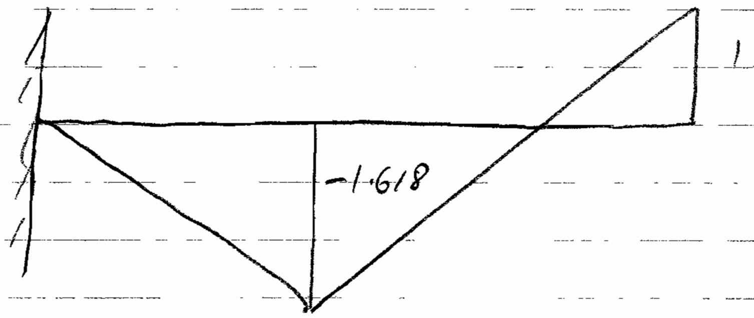
$$@\ \omega_1^2 = 0.255\ k/m, \quad y_1/y_2^{(1)} = \frac{1}{2 - \frac{3}{2}(0.255)} = 0.618$$

$$@\ \omega_2^2 = 1.745\ k/m \quad y_1/y_2^{(2)} = \frac{1}{2 - \frac{3}{2}(1.745)} = -1.618$$





$@\ \omega_1^2 = 0.255\,k/m$



$@\ \omega_2^2 = 1.745\,k/m$

(c) Transient response

$$\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} (A \cos \omega_1 t + B \sin \omega_1 t) + \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} (C \cos \omega_2 t + D \sin \omega_2 t)$$

Initial Conds $\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} R\pi/4 \\ 0 \end{Bmatrix}$ @ $t = 0$.

Initial displacement $\Rightarrow$ Sin terms are zero $B = D = 0$.

3. (cont)

$$\begin{Bmatrix} R\pi/4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} A + \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} C$$

$$\Rightarrow \quad R\pi/4 = 0.618A - 1.618C$$

$$0 = A + C \quad \Rightarrow \quad A = -C$$

So $R\pi/4 = 0.618A + 1.618A \quad \Rightarrow \quad A = 0.351R$

$C = -0.351R$

so $y_1 = 0.217R\cos\omega_1 t + 0.568R\cos\omega_2 t$

& $y_2 = 0.351R\cos\omega_1 t - 0.351R\cos\omega_2 t$

check $t=0$, $y_1 = 0.785R = \pi R/4$ ✓

$y_2 = 0$ ✓

At $t = \sqrt{\frac{m}{k}}$, $y_1 = 0.217R\cos\sqrt{0.255}$

$+ 0.568R\cos\sqrt{1.745}$

& $\theta_1 = y_1/R = \underline{\underline{0.33}}$ radians $(18.9°)$

(d) k_2 is increased by 20%
Calculate new ω's using Rayleigh's quotient with old mode shapes.

Rayleigh: $\omega^2 = \frac{V_{max}}{T^*} = \frac{\cancel{\tfrac{1}{2}} k y_1^2 + \cancel{\tfrac{1}{2}}(1.2k)(y_2 - y_1)^2}{\cancel{\tfrac{1}{2}}\, \tfrac{3}{2}m\,(y_1^2 + y_2^2)}$

$$= \frac{2}{3}\,k/m\left[\frac{y_1^2 + 1.2(y_2-y_1)^2}{y_1^2+y_2^2}\right]$$

$\begin{Bmatrix} y_1^{(1)} \\ y_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix} \Rightarrow \omega_1^2 \simeq \frac{2}{3}\frac{k}{m}\left(\frac{0.618^2 + 1.2(0.382)^2}{0.618^2 + 1^2}\right) = \underline{\underline{0.269\frac{k}{m}}}$ (higher as expected)

3. Cont $\begin{Bmatrix} y_1^{(2)} \\ y_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} -1.618 \\ 1 \end{Bmatrix} \Rightarrow \omega_2^2 \simeq \frac{2}{3}\frac{k}{m}\left(\frac{1.618^2 + 1.2\,(2.618)^2}{1.618^2 + 1^2}\right)$

$$\underline{\underline{\omega_2^2 = 2.0\, k/m}}$$

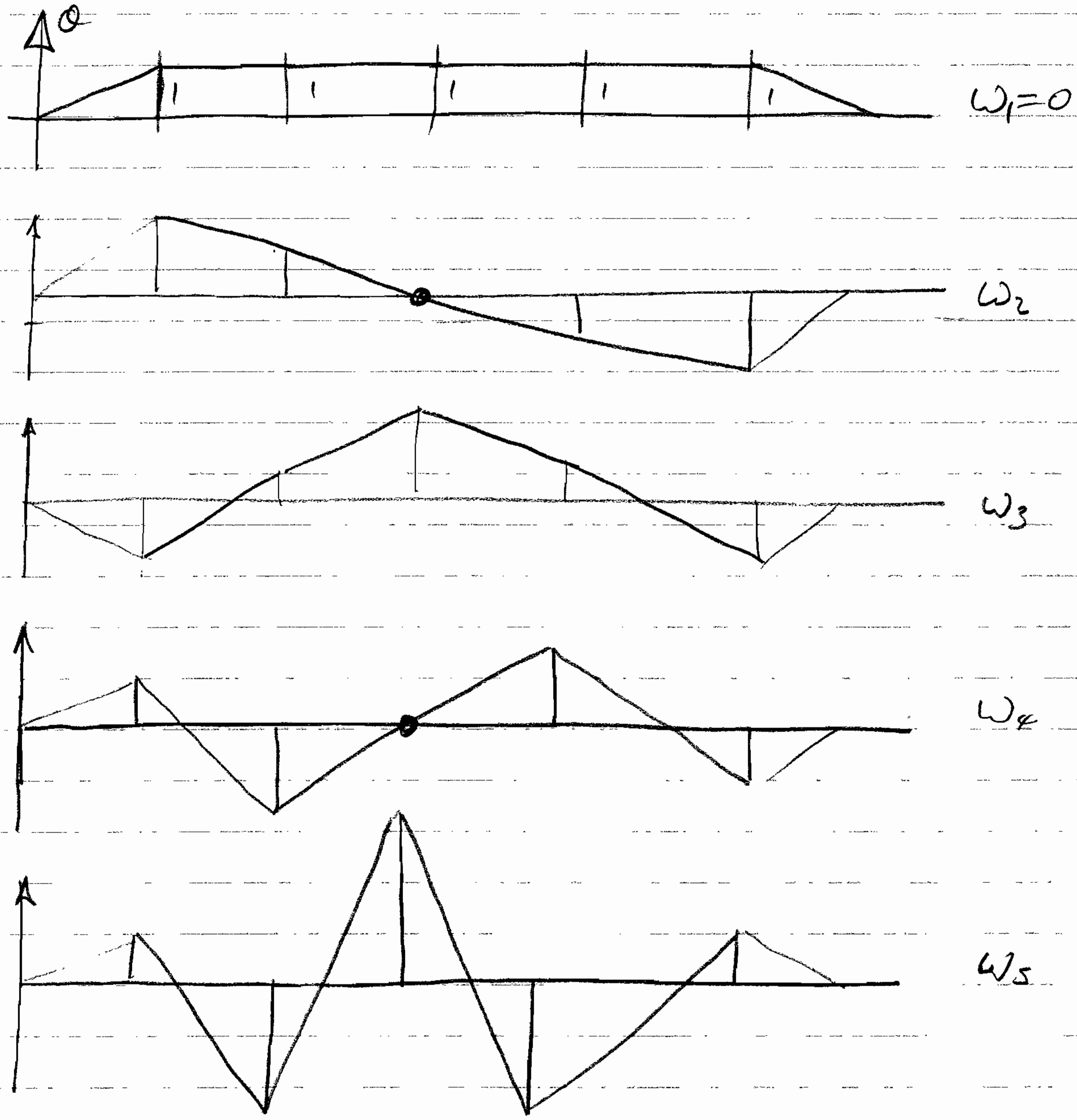
At $t = \sqrt{m/k}$, $\theta_1 \approx 0.217 \cos\sqrt{0.269} + 0.568 \cos\sqrt{2.0}$
$= 0.277$ <u>rads</u>

Q4

(a) $$T = \tfrac{1}{2} J \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2 + \dot{\theta}_5^2 \right)$$

$$V = \tfrac{1}{2} k \left[(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + (\theta_4 - \theta_3)^2 + (\theta_5 - \theta_4)^2 \right]$$

(b)



Modes are either symmetric or antisymmetric

(c) Assume $\underline{q} = [1 \quad 0.5 \quad 0 \quad -0.5 \quad -1]$

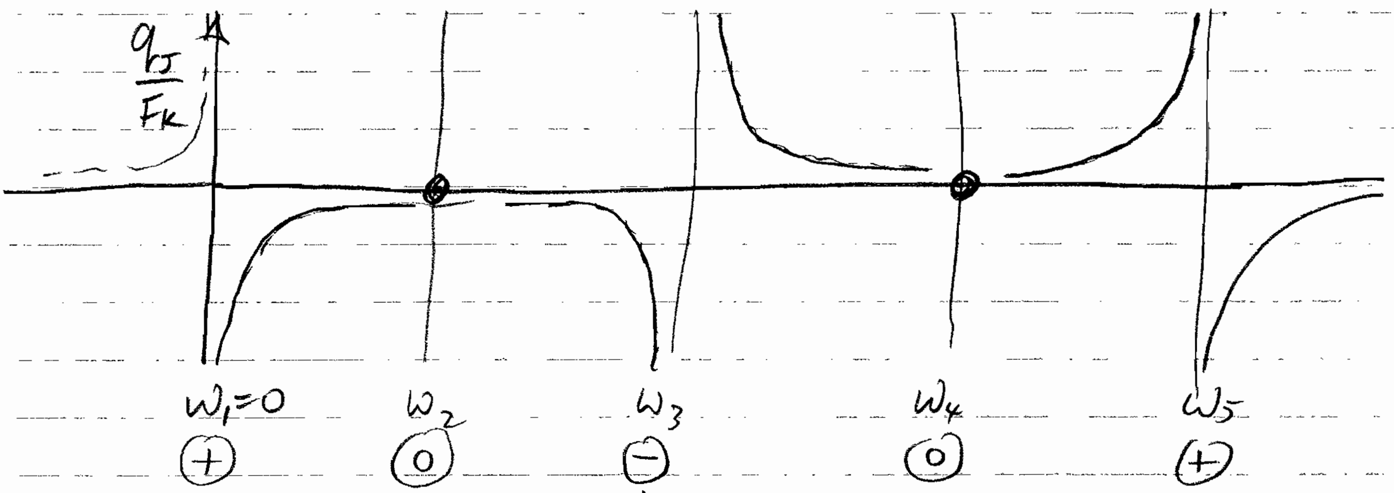
Rayleigh: $\omega^2 \approx \dfrac{V_{max}}{T^*}$

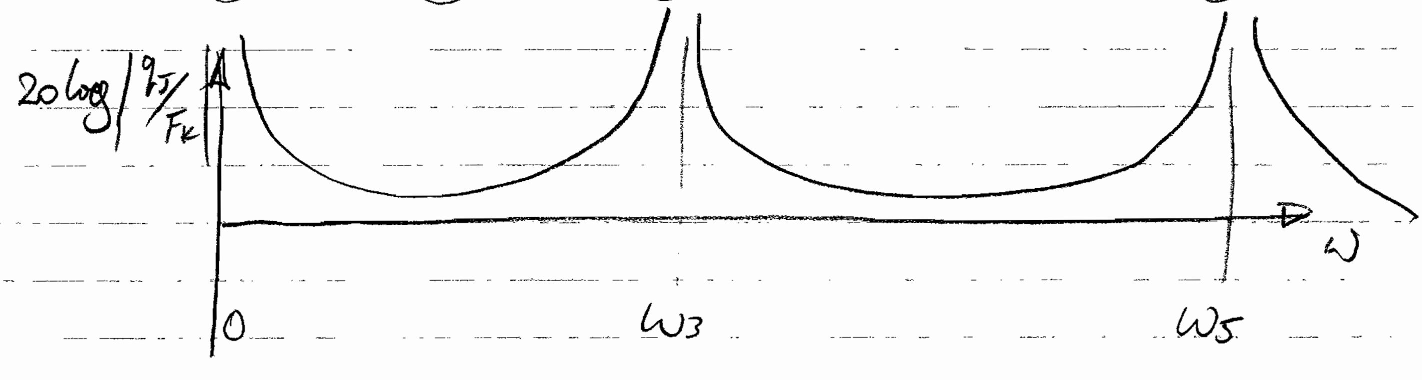
ie $\omega^2 \simeq \dfrac{\frac{1}{2}k\left[(0.5-1)^2 + (0-0.5)^2 + (-0.5-0)^2 + (-1+0.5)^2\right]}{\frac{1}{2}J\left[1^2 + 0.5^2 + 0^2 + (-0.5)^2 + (-1)^2\right]}$

$= \dfrac{k}{J}\left(\dfrac{4(0.5)^2}{2(0.5^2)+2}\right) = \underline{\underline{0.4\,k/J}} \qquad \left(\begin{matrix}0.382\,k/J \\ \text{exact}\end{matrix}\right)$

(d) $\dfrac{q_J}{F_K} = \sum_N \dfrac{u_J^{(n)} u_K^{(n)}}{\omega_n^2 - \omega^2}$ with $J=3,\ K=1$

Signs of $u_J^{(n)} u_K^{(n)}$ are: $[+,\ 0,\ -,\ 0,\ +]$





(d) Flywheel has different moment of inertia, no damping, shaft is not massless, etc

ENGINEERING TRIPOS PART IIB

Module 3C6 Examination, 2006

Answers

1 (b) $x = 0,\ \theta = 0;\ x = L,\ d\theta/dx = 0.\quad \omega_n = \sqrt{\dfrac{G}{\rho}}\left(n+\dfrac{1}{2}\right)\dfrac{\pi}{L};\quad u_n(x) = \sin\left(n+\dfrac{1}{2}\right)\dfrac{\pi x}{L}$

(c) $K\dfrac{\partial^2\theta}{\partial t^2} = -GJ\dfrac{\partial\theta}{\partial x}$ at $x = L$; $\tan kL = \dfrac{J\rho}{Kk}$, with $k^2 = \dfrac{\rho\omega^2}{G}$

2 (b) $\omega_n^2 \approx \left(n+\dfrac{1}{2}\right)^4 \pi^4 \dfrac{EL}{mL^4}$; (c) $\omega^2 = 504\dfrac{EI}{mL^4}$

3 (a) $T = \dfrac{1}{2}\left(\dfrac{3}{2}m\dot{y}_1^2 + \dfrac{3}{2}m\dot{y}_2^2\right)$; $V = \dfrac{1}{2}ky_1^2 + \dfrac{1}{2}k\left(y_2 - y_1\right)^2$; $\dfrac{3m}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$; $k\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$

(b) $\omega^2 = \dfrac{k}{m}\left(1 \pm \dfrac{\sqrt{5}}{3}\right)$; $\begin{bmatrix}0.618 & 1\end{bmatrix}^T$; $\begin{bmatrix}-1.618 & 1\end{bmatrix}^T$

(c) 0.33 rads; (d) 0.28 rads

4 (a) $T = \dfrac{1}{2}J\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2 + \dot{\theta}_5^2\right)$

$$V = \frac{1}{2}k\left[\left(\theta_2-\theta_1\right)^2 + \left(\theta_2-\theta_1\right)^2 + \left(\theta_3-\theta_2\right)^2 + \left(\theta_4-\theta_3\right)^2 + \left(\theta_5-\theta_4\right)^2\right]$$

(c) $\omega_1^2 \approx 0.4\dfrac{k}{J}$