PART IIA 2006
3C7: Mechanics of solids
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1. (a) For a plane problem with circular symmetry, the following conditions hold:

① The stress state is symmetric about any diameter

② Shear stress $\sigma_{r\theta} = 0$ along any radial line

③ Stresses σ_{rr} and $\sigma_{\theta\theta}$ are functions of r only

(b) From data sheet, the general expressions for equilibrium equations are

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0 \qquad (i)$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0 \qquad (ii)$$

For a circular disk with circular symmetry, eqn (ii) is automat. satisfied, whereas eqn (i) becomes

$$\frac{\partial}{\partial r}(r\sigma_{rr}) - \sigma_{\theta\theta} = 0$$

Rearranging, we get $$r\frac{d\sigma_{rr}}{dr} = \sigma_{\theta\theta} - \sigma_{rr}$$

(c) The compatability eqn is (data sheet)

$$\frac{\partial}{\partial r}\left\{r\frac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \frac{\partial}{\partial r}\left\{r^2\frac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\frac{\partial \varepsilon_{rr}}{\partial r} + \frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$$

For our thin circular disk, since $\gamma_{r\theta} = 0$ and ε_{rr}, $\varepsilon_{\theta\theta}$ are independent of θ, this simplifies to

1.(c) continued

$$\frac{\partial}{\partial r}\left\{ r^2 \frac{\partial \varepsilon_{\theta\theta}}{\partial r} \right\} - r \frac{\partial \varepsilon_{rr}}{\partial r} = 0$$

$$\Rightarrow \quad r^{\cancel{2}} \frac{d^2 \varepsilon_{\theta\theta}}{dr^2} + 2\cancel{r} \frac{d\varepsilon_{\theta\theta}}{dr} - \cancel{r} \frac{d\varepsilon_{rr}}{dr} = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r \frac{d\varepsilon_{\theta\theta}}{dr} \right) - \frac{d\varepsilon_{\theta\theta}}{dr} + 2 \frac{d\varepsilon_{\theta\theta}}{dr} - \frac{d\varepsilon_{rr}}{dr} = 0$$

$$\Rightarrow \quad \frac{d}{dr}\left(r \frac{d\varepsilon_{\theta\theta}}{dr} \right) = \frac{d(\varepsilon_{rr} - \varepsilon_{\theta\theta})}{dr}$$

$$\Rightarrow \quad \boxed{r \frac{d\varepsilon_{\theta\theta}}{dr} = \varepsilon_{rr} - \varepsilon_{\theta\theta}}$$

(d)

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \nu \frac{\sigma_{\theta\theta}}{E} + \alpha (T - T_0)$$

$$\varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu \frac{\sigma_{rr}}{E} + \alpha (T - T_0)$$

$$\gamma_{r\theta} = \frac{\sigma_{r\theta}}{G}, \qquad G = \frac{E}{2(1+\nu)}$$

(e)

$$\frac{d}{dr}\left[\frac{1}{r} \frac{d(ur)}{dr} \right] = (1+\nu)\,\alpha \frac{d(T-T_0)}{dr}$$

Integration once $\Rightarrow \quad \frac{1}{r}\frac{d(ur)}{dr} = (1+\nu)\,\alpha (T-T_0) + 2C_2$

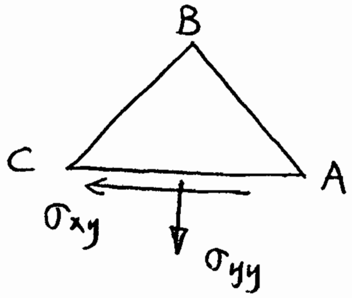
$$\Rightarrow \quad \frac{d(ur)}{dr} = (1+\nu)\,\alpha \Delta T\, r + 2C_2 r$$

where $\Delta T \equiv T(r) - T_0$

Integrate again $\Rightarrow \quad ur = C_2 r^2 + C_1 + (1+\nu)\,\alpha \int_a^r r \Delta T\, dr$

$$\Rightarrow \quad u = C_2 r + \frac{C_1}{r} + (1+\nu) \frac{\alpha}{r} \int_a^r r \Delta T\, dr$$

2. (a) Let's analyse the free body diagram of extrusion AB.



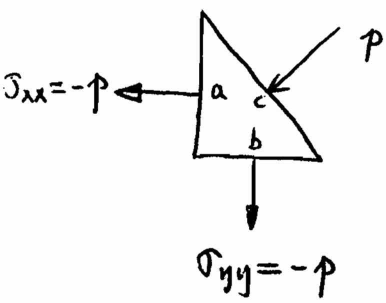
Because both AB and BC surfaces are stress free, hence $\sigma_{yy} = \sigma_{xy} = 0$ on AC. Therefore, the whole extrusion is free of stress.

(b) (i) Equilibrium equations (from data sheet):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0$$

Obviously, the stress state $\sigma_{xx} = \sigma_{yy} = -p$, $\sigma_{xy} = 0$ satisfies the above equations.

Along surface CDEFG, it is obvious that the pressure p is balanced by either σ_{xx} or σ_{yy}. Let's analyse the situation along AB.



Force balance in x-direction gives

$$\sigma_{xx} \cdot \cancel{a} + pc \times \frac{\cancel{a}}{\cancel{c}} = 0 \Rightarrow -p + p = 0$$

In the y-direction

$$\sigma_{yy} \cdot b + pc \times \frac{b}{c} = 0 \Rightarrow -p + p = 0$$

Similarly, it can be shown that the boundary condition on surface BC is satisfied.

2. (b) continued

(ii)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\sigma_{xx}}{E} - \frac{\nu \sigma_{yy}}{E} = \frac{\nu - 1}{E} p$$

$$\Rightarrow \quad u = \frac{\nu-1}{E} p x + c_1(y)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\sigma_{yy}}{E} - \frac{\nu \sigma_{xx}}{E} = \frac{\nu - 1}{E} p$$

$$\Rightarrow \quad v = \frac{\nu-1}{E} p y + c_2(x)$$

But $\sigma_{xy} = 0 \Rightarrow \gamma_{xy} = 0$

$$\Rightarrow \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{d c_1(y)}{dy} + \frac{d c_2(x)}{dx} = 0$$

$$\Rightarrow \quad \frac{dc_1}{dy} = -\frac{dc_2}{dx} = A = \text{constant}$$

$$\Rightarrow \quad c_1(y) = Ay + B, \qquad c_2(x) = -Ax + C$$

$$\Rightarrow \quad u = \frac{\nu-1}{E} p x + Ay + B, \quad v = \frac{\nu-1}{E} p y - Ax + C$$

(iii) $\varepsilon_{xx} = \frac{\nu-1}{E} p$, $\varepsilon_{yy} = \frac{\nu-1}{E} p$, $\gamma_{xy} = 0$

Compatability:
$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$$

This is satisfied.

(iv) The stress field $\sigma_{xx} = \sigma_{yy} = -p$, $\sigma_{xy} = 0$ satisfies equilibrium and boundary conditions (including the surface of hole). Also, the resulting strain field satisfies compatability. In other words, the solutions of (i) and (ii) remain valid.

Q3

(a) Airy stress function if $\nabla^4\phi = 0 = 2Ay + 6Dy$

$$\Rightarrow \quad D = -\frac{2A}{3} \qquad (15\%)$$

(b) $\sigma_{xx} = A\left(x^2y - \frac{2y^3}{3}\right), \quad \sigma_{yy} = \frac{Ay^3}{3} + 2Bxy + Cy$

$\sigma_{xy} = -Axy^2 - Bx^2 - Cx$

Boundary conditions

$\sigma_{xy}(x, \pm b) = 0 = -Axb^2 - Bx^2 - Cx$

$-\sigma_{yy}(x,-b) = q = +\frac{Ab^3}{3} + 2Bxb + Cb$

$\sigma_{yy}(x,+b) = q = \frac{Ab^3}{3} + 2Bxb + Cb$

$\Rightarrow B = 0, \quad C = Ab^2, \quad A = \frac{3q}{2b^3}$

Stresses

$\sigma_{xx} = \frac{3q}{2b^3}\left(x^2y - \frac{2y^3}{3}\right); \quad \sigma_{yy} = \frac{3q}{2b^3}\left(\frac{y^3}{3} - b^2\right.$

$\sigma_{xy} = -\frac{3q}{2b^3}\left(b^2x - xy^2\right) \qquad (40\%$

(c) $S(x) = \int_{-b}^{b} \sigma_{xy}(x,y)\,dy = -2q\cancel{x}$ ✓ $\qquad (20\%$

(d)

$$M(0) = \int_{-b}^{b} \sigma_{xx}(0, y)\, y\, dy = \frac{3q}{\cancel{2}b^3}\int_{-b}^{b} \frac{y^4}{3}\, dy$$

$$= \frac{2b^2 q}{5}$$

Thus add ~~a~~ a constant bending moment to correct bending moment distribution. This only modifies σ_{xx}

$$\sigma_{xx} = \frac{3q}{2b^3}\left(x^2 y - \frac{2y^3}{3}\right) - \frac{2b^2 q\, y}{5} \cdot \overset{I}{\overset{\swarrow}{\left(\frac{3}{2b^3}\right)}}$$

$$= \frac{3q}{2b^3}\left(x^2 y - \frac{2y^3}{3} - \frac{2b^2 y}{5}\right)$$

(25%)

Q 4 (a) book - work (30%)

(b)

(i) Work done by applied pressure (unit thickness)

$$= (2\pi a)\, p \frac{A}{a} = 2\pi p A$$

(20%)

(ii) $d\varepsilon_{rr} = \frac{\partial u}{\partial r} = -\frac{A}{r^2}$

$d\varepsilon_{\theta\theta} = \frac{A}{r^2}$, $d\varepsilon_{zz} = 0$ (plane strain)

$d\varepsilon_{rr} : d\varepsilon_{\theta\theta} : d\varepsilon_{zz} = -1 : 1 : 0$ (satisfies Tresca flow rule

$$dw = \sigma_{\theta\theta}\, d\varepsilon_{\theta\theta} + \sigma_{rr}\, d\varepsilon_{rr} + \sigma_{zz}\, d\varepsilon_{zz}$$

Active Tresca criterion $\sigma_{\theta\theta} - \sigma_{rr} = Y$

$$\Rightarrow dw = (\sigma_{\theta\theta} - \sigma_{rr}) \frac{A}{r^2} = \frac{YA}{r^2}$$

$$W = 8 \int_0^{\pi/4} \int_a^{\frac{b}{\cos\theta}} \frac{YA}{r^2}\, r\, dr\, d\theta$$

$$= 8YA \int_0^{\pi/4} \ln\left(\frac{b}{a\cos\theta}\right) d\theta$$

$$= 8YA \left[\frac{\pi}{4} \ln\left(\frac{b}{a}\right) - \int_0^{\pi/4} \ln\cos\theta\, d\theta \right]$$

$$W = \cancel{8} \; 8 \Upsilon A \left[\frac{\pi}{4} \ln \frac{b}{a} + \frac{\pi}{4} \ln 2 - 0.458 \right]$$

$$p_c \, p A \, 2\pi = 8 \Upsilon A \left[\frac{\pi}{4} \ln \frac{2b}{a} - 0.458 \right]$$

$$p_c = \cancel{8\Upsilon} \; \frac{4 \Upsilon}{\pi} \left(\frac{\pi}{4} \ln \frac{2b}{a} - 0.458 \right)$$

(40%)

(iii) The assumed displacement field is axi-symmetric while the problem is not. It is thus likely that the collapse pressure estimated above is substantially al[…] the "actual" collapse pressure.

Answers to 3C7: Mechanics of Solids (2005-2006)

1. (e) $u = C_2 r + \dfrac{C_1}{r} + (1+\nu)\dfrac{\alpha}{r}\int_a^r r\Delta T dr$

3. (a) $D = -\dfrac{2A}{3}$

4. (b)(i) $W = 2\pi pA$

(b)(ii) $p_c = \dfrac{4Y}{\pi}\left(\dfrac{\pi}{4}\ln\dfrac{2b}{a} - 0.458\right)$

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