Module 3F1, April 2006 – SIGNALS AND SYSTEMS – Solutions

ENGINEERING TRIPOS PART IIA

Tuesday 2 May 2006 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Module 3F1, April 2006 – SIGNALS AND SYSTEMS – Answers

1 (a) Open-loop poles at $-1/2 \pm j\sqrt{7}/2$. Unstable.

(b) $T(z) = -k/(z^2+z+2-k)$.

(c) $1 < k < 2$.

(d) For $1 < k < 2$, $\lim_{n\to\infty} y_n = -k/(4-k)$.

(e) B.

2 (a) (i) $g_n = (1,-1,1,-2,1,0,0,...)$.

(ii) Causal and stable.

(iii) Input $= u_n - 2u_{n-1}$.

(b) $g(x) = \begin{cases} \ln(x+1) & \text{for } -1 < x < 0 \\ -\ln(1-x) & \text{for } 0 \leq x < 1 \end{cases}$

3 (a) $S_X(\omega) = \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau} d\tau$

(b) $\omega_c = 2/T_0$.

(c) $|H(\omega)| = \begin{cases} \sqrt{\frac{B}{\pi P T_0} \frac{1-|\omega|T_0}{1-\frac{1}{2}|\omega|T_0}} & \text{if } |\omega| < 1/T_0 \\ 0 & \text{if } 1/T_0 \leq |\omega| < 2/T_0 \end{cases}$

(d) $h(t) = \mathrm{FT}^{-1}\left(\frac{S_{XY}(\omega)}{S_X(\omega)}\right)$.

4 (a)

X_t \ X_{t+1}	H	T
H	0.9	0.1
T	0.5	0.5

$P_H = 5/6$ and $P_T = 1/6$.

(b) $H(X_{t+1}, X_t) = 1.2075$, $H(X_{t+2}, X_t) = 1.2837$.

(c) C(Code)-0 S(Symbol)-HH, C-10 S-HT, C-110 S-TH, C-111 S-TT. $L/2 = 0.7082$.

(e) No.

1 (a) The open–loop system has two poles at $-1/2 \pm j\sqrt{7}/2$. The pole-zero diagram in Fig. 3 shows all the poles outside the unit disk, and so the system is unstable. [20%]

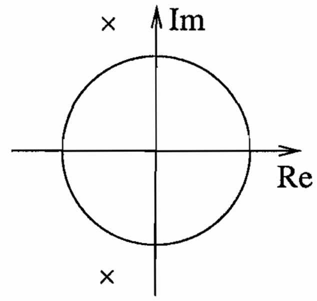

Fig. 3

(b) The closed–loop transfer function is given by

$$T(z) = \frac{Y(z)}{R(z)} = \frac{kP(z)}{1+kP(z)} = \frac{-k}{z^2+z+2-k}$$

[10%]

(c) The closed loop poles are the roots of $z^2+z+2-k=0$ which are given by

$$z = \frac{-1 \pm \sqrt{4k-7}}{2}$$

For closed–loop stability we need the poles inside of the unit disk. For $4k-7<0$ the poles are complex and therefore we need

$$\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{4k-7}}{2}\right)^2 < 1$$

or $|4k-7|<3$. Since $7-4k>0$, we get $7-4k<3$, or $1<k<7/4$. For $4k-7 \geq 0$ the poles are real and therefore we need

$$-1 < \frac{-1 \pm \sqrt{4k-7}}{2} < 1$$

or $-2 < -1-\sqrt{4k-7}$ and $-1+\sqrt{4k-7} < 2$, which is equivalent to $\sqrt{4k-7} < 1$ and $\sqrt{4k-7} < 3$. Since $4k-7 \geq 0$, we get $7/4 \leq k < 2$. Combining both, we get the final answer of $1<k<2$. [30%]

(cont.

(d) When $k \notin (1,2)$ the system is unstable and therefore y_n will grow unbounded. When $k \in (1,2)$, the closed–loop system is stable and we use the final value theorem (the closed–loop transfer function $T(z)$ was found in part (b)):

$$\lim_{n\to\infty} y_n = \lim_{z\to 1}(z-1)\frac{-k}{z^2+z+2-k}\frac{z}{z-1} = \frac{-k}{4-k}$$

[20%]

(e) For $k = 1.5$, the closed–loop system is stable. Because the open–loop system has two unstable poles, the Nyquist plot must have two conterclockwise encirclements of the -1 point. Thus, plot B is the correct one.

Alternatively, the open–loop system is

$$kP(z) = \frac{-3/2}{z^2+z+2}$$

Nyquist plot A shows that for $z = 1$ or $z = -1$ $kP(z) = -1.5$. However, $kP(1) = -3/8$ and $kP(-1) = -3/4$, thus plot A cannot be the one. Nyquist plot C shows that the magnitude of $kP(z)$ is always approximately less 0.75. However, $|kP(e^{j1.93})| = 1.6$. Also, there exists only one encirclement and so the system could never be stable. Thus, plot C cannot be the one. Plot B satisfies all of the above conditions and is the correct one. In addition, the closed–loop system is stable as explained above. [20%]

2 (a) (i) Since the pulse is the difference between elements of the step, the reponse of the pulse is also the difference of the response of the step. Thus,

$$g_n = \begin{cases} 0 & n \leq -1 \\ 1 & n = 0 \\ -1 & n = 1 \\ 1 & n = 2 \\ -2 & n = 3 \\ 1 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

Alternatively, we could have found the transfer function first:

$$Y(z) = 1 + z^{-2} - z^{-3} = G(z)\frac{1}{1 - z^{-1}}$$

Thus,

$$G(z) = 1 - z^{-1} + z^{-2} - 2z^{-3} + z^{-4}$$

The inverse z–transform gives the above result. [20%]

(ii) The system is causal since it responds at the same time (and not before) the input. It is also stable since $\sum_{k=0}^{\infty} |g_k|$ is finite. [10%]

(iii) Since we know the response of the system to a step, we can use linearity and time–invariant properties to find the answer. At time $n = 0$ we need $x_0 = 1$. That can be obtained with the step u_n. To obtain $x_1 = -2$, we need to add to u_n the term $-2u_{n-1}$. It turns out that the input $u_n - 2u_{n-1}$ results in the response we are looking for. [20%]

(b) (i) When $g(.)$ is monotonic increasing, the cdf of Y is given by

$$F_Y(y) = \Pr\{Y \leq y\} = \Pr\{g(X) \leq g(x)\} = \Pr\{X \leq x\} = F_X(x)$$

since $Y = g(X)$ and $y = g(x)$, and $g(X) \leq g(x)$ implies that $X \leq x$ if $g(.)$ is monotonic increasing. [10%]

(ii) Now $f_X(x) = \frac{1}{2}$ over the range $-1 < x < 1$, to make the uniform pdf valid. So the cdf of X will be

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = \begin{cases} 0 & \text{for } x < -1 \\ (x+1)/2 & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(cont.

The cdf of Y will be

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)\,du = \int_{-\infty}^{y} \frac{1}{2}\exp(-|y|)\,du$$

To deal with the modulus of y, we consider $y < 0$ and $y \geq 0$ separately.
If $y < 0$:

$$F_Y(y) = \int_{-\infty}^{y} \frac{1}{2}\exp(y)\,du = \left[\frac{1}{2}\exp(y)\right]_{-\infty}^{y} = \frac{1}{2}\exp(y)$$

If $y \geq 0$:

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{0} \frac{1}{2}\exp(y)\,du + \int_{0}^{y} \frac{1}{2}\exp(-y)\,du \\ &= \frac{1}{2} + \left[-\frac{1}{2}\exp(-y)\right]_0^y = 1 - \frac{1}{2}\exp(-y) \end{aligned}$$

Using the result of part (i):

$$F_Y(y) = F_X(x) = \frac{x+1}{2}$$

Therefore if $y < 0$:

$$\frac{x+1}{2} = \frac{1}{2}\exp(y) \quad \text{and so} \quad y = \ln(x+1)$$

and if $y \geq 0$:

$$\frac{x+1}{2} = 1 - \frac{1}{2}\exp(-y) \quad \text{and so} \quad y = -\ln(1-x)$$

We note that if $-\infty < y < 0$ then $-1 < x < 0$, and if $0 \leq y < \infty$ then $0 \leq x < 1$, so $g(.)$ is given by:

$$y = g(x) = \begin{cases} \ln(x+1) & \text{for } -1 < x < 0 \\ -\ln(1-x) & \text{for } 0 \leq x < 1 \end{cases}$$

and $g(.)$ correctly maps the full range of X to the full range of Y. A check on the gradient of g reveals that it is indeed strictly monotonically increasing, as required. [40%]

(TURN OVER

3 (a) The *Power Spectral Density* (PSD) of a random process X is the Fourier Transform of its ACF:

$$S_X(\omega) = \text{FT}\{r_{XX}(\tau)\} = \int_{-\infty}^{\infty} r_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

N.B. $\{X(t)\}$ must be *at least* Wide Sense Stationary (WSS) for this to be valid. In this case it is ergodic which implies that it is stationary. [15%]

(b) We need to calculate the Fourier transform of the given $\text{sinc}^2(.)$ function.

From the E&I data book page 25, a triangular pulse $g(t)$ of base width $2b$ and height A transforms to the function $Ab\ \text{sinc}^2(\omega b/2)$ in the Fourier domain. Hence:

$$Ab\ \text{sinc}^2(\omega b/2) = \int_{-\infty}^{\infty} g(t) \exp(-j\omega t)\, dt$$

We may interchange t and ω and change t to τ to get:

$$Ab\ \text{sinc}^2(\tau b/2) = \int_{-\infty}^{\infty} g(\omega) \exp(-j\omega\tau)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi g(\omega) \exp(j\omega\tau)\, dt$$

since g is an even function and the LHS is purely real.

Hence $2\pi g(\omega)$ is the Fourier transform of $Ab\ \text{sinc}^2(\tau b/2)$, and the given ACF will transform to a triangular-shaped PSD, whose half-width b and amplitude $2\pi A$ are given by:

$$b = \frac{2}{T_0} \quad \text{and} \quad 2\pi A = \frac{2\pi P}{b} = \pi P T_0$$

So, expressing the triangular PSD pulse in full:

$$S_X(\omega) = \begin{cases} \pi P T_0 (1 - \frac{1}{2}|\omega| T_0) & \text{if } |\omega| < 2/T_0 \\ 0 & \text{if } |\omega| \geq 2/T_0 \end{cases}$$

Hence, $\omega_c = 2/T_0$. [30%]

(c) For a linear system, excited by a signal with PSD S_X and producing an output signal with PSD S_Y, the power gain is $|H(\omega)|^2$. Hence

$$S_Y(\omega) = S_X(\omega)\ |H(\omega)|^2 \quad \text{and so} \quad |H(\omega)| = \sqrt{\frac{S_Y(\omega)}{S_X(\omega)}}$$

Hence, as long as $|S_X(\omega)| > 0$, we can calculate $|H(\omega)|$. Therefore if $|\omega| < 2/T_0$

$$|H(\omega)| = \sqrt{\frac{S_Y(\omega)}{S_X(\omega)}} = \begin{cases} \sqrt{\frac{B}{\pi P T_0} \frac{1 - |\omega| T_0}{1 - \frac{1}{2}|\omega| T_0}} & \text{if } |\omega| < 1/T_0 \\ 0 & \text{if } 1/T_0 \leq |\omega| < 2/T_0 \end{cases}$$

(cont.

It is not feasible to calculate $|H(\omega)|$ outside the range $|\omega| < 2/T_0$ because both the numerator and denominator in the ratio $\frac{S_Y(\omega)}{S_X(\omega)}$ will be zero, leading to an indeterminate answer. [30%]

(d) It is difficult to calculate $h(t)$, which is the inverse Fourier transform of $H(\omega)$, because the above method only gives us $|H(\omega)|$ and there is no information on the phase component of $H(\omega)$. We need both amplitude and phase information about a function in order to perform an inverse Fourier transform on it.

To overcome this difficulty it is necessary to measure the Cross Spectral Density $S_{XY}(\omega)$, which is the Fourier transform of the Cross-Correlation Function $r_{XY}(\tau)$ between X and Y. We find that

$$S_{XY}(\omega) = S_X(\omega)\, H(\omega) \quad \text{and hence} \quad H(\omega) = \frac{S_{XY}(\omega)}{S_X(\omega)}$$

which gives both phase and amplitude information about $H(\omega)$, as required. Hence

$$h(t) = \mathrm{FT}^{-1}\left(\frac{S_{XY}(\omega)}{S_X(\omega)}\right)$$

[25%]

(TURN OVER

4 (a) The table of conditional probabilities of X_{t+1} given X_t is

X_t \ X_{t+1}	H	T
H	0.9	0.1
T	0.5	0.5

Let $P_H = \text{Prob}\{X_t = \text{H}\}$ and $P_T = \text{Prob}\{X_t = \text{T}\}$ then the equilibrium condition is

$$P_H = 0.9P_H + 0.5P_T$$
$$P_H = 1 - P_T\,.$$

Hence, the equilibrium distribution of X_t is $P_H = 5/6$ and $P_T = 1/6$. [20%]

(b) We have $H(X_t) = -(5/6)\log_2(5/6) - (1/6)\log_2(1/6) = 0.65$ and

$$H(X_{t+1}|X_t = \text{H}) = -0.9\log_2(0.9) - 0.1\log_2(0.1) = 0.4690$$
$$H(X_{t+1}|X_t = \text{T}) = -0.5\log_2(0.5) - 0.5\log_2(0.5) = 1$$
$$H(X_{t+1}|X_t) = (5/6)\cdot 0.469 + (1/6)\cdot 1 = 0.5575\,.$$

Hence, $H(X_{t+1}, X_t) = H(X_t) + H(X_{t+1}|X_t) = 1.2075$.

The table of conditional probabilities of X_{t+2} given X_t is:

X_t \ X_{t+2}	H	T
H	0.86	0.14
T	0.7	0.3

Therefore we obtain

$$H(X_{t+2}|X_t = \text{H}) = -0.86\log_2(0.86) - 0.14\log_2(0.14) = 0.5842$$
$$H(X_{t+2}|X_t = \text{T}) = -0.7\log_2(0.7) - 0.3\log_2(0.3) = 0.8813$$
$$H(X_{t+2}|X_t) = (5/6)\cdot 0.58424 + (1/6)\cdot 0.8813 = 0.6337\,.$$

Hence, $H(X_{t+2}, X_t) = H(X_t) + H(X_{t+2}|X_t) = 1.2837$.

The fact that $H(X_{t+2}, X_t) > H(X_{t+1}, X_t)$ means that the mutual information between X_{t+2} and X_t is less than the mutual information between X_{t+1} and X_t. [25%]

(cont.

(c) A Huffman code is:

```
Code Symbol Probability

                                                    0
0       HH   0.75-------------------------------\
                                       0          \
10      HT   0.0833-----------------\           1  --1
                     0                 --0.25------/
110     TH   0.0833--\            1 /
                     1  ---0.1666--/
111     TT   0.0833--/
```

The average codeword length is $L = 1.4164$. Hence the average number of bits per symbol is $L/2 = 0.7082$. [25%]

(d) The joint entropy of N symbols is

$$H(X_{t+N-1},\ldots,X_{t+1},X_t) = H(X_t) + (N-1)H(X_{t+1}|X_t) = 0.65 + (N-1)0.5575\,.$$

Therefore the theoretical lower bound on the average number of bits per symbol is

$$\frac{1}{N}H(X_{t+N-1},\ldots,X_{t+1},X_t) = 0.5575 + \frac{1}{N}0.0925\,.$$

For $N \geq 50$ we have $\frac{1}{N}H(X_{t+N-1},\ldots,X_{t+1},X_t) \leq 0.56$. Hence an average of 0.6 bits per original symbol does not violate the entropy bound. [15%]

(e) It is always possible to construct a code with an average number of bits per symbols less than

$$\frac{1}{N}H(X_{t+N-1},\ldots,X_{t+1},X_t) + \frac{1}{N}\,.$$

For $N \geq 50$ we have $\frac{1}{N}H(X_{t+N-1},\ldots,X_{t+1},X_t) + \frac{1}{N} \leq 0.58$. Therefore, a code which encodes more than 50 symbols at a time with 0.6 bits per symbol in average is not optimal. Hence, the claim cannot be correct. [15%]

END OF PAPER