PART IIA 2006
3F2: Systems and control
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Datasheet: **None**

3F2 Systems and Control: 2006 Solutions

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1. (a) Taking Laplace transforms:

$$s\bar{x}(s) - x(0) = A\bar{x}(s) + B\bar{u}(s)$$

Assume that $x(0) = 0$. Then

$$\bar{x}(s) = (sI - A)^{-1}B\bar{u}(s)$$

But $\bar{y}(s) = C\bar{x}(s) + D\bar{u}(s)$, so

$$\bar{y}(s) = C(sI - A)^{-1}B\bar{u}(s) + D\bar{u}(s)$$

so the transfer function matrix is

$$G(s) = C(sI - A)^{-1}B + D$$

(b) At steady-state we have $\dot{H} = 0$. Hence $M^* = \sqrt{P^* + \frac{1}{5}}$. But $\sqrt{P^* + \frac{1}{5}} = M_o = 1$ at steady-state. So $M^* = 1$ and $P^* = \frac{4}{5}$.

(c) Take a Taylor series expansion up to first-order terms:

$$\begin{aligned} \dot{H} = \dot{x} &= (M^* + u_1) - \sqrt{(P^* + u_2) + \frac{1}{5}(1 + x)} \triangleq f(x, u_1, u_2) \\ &\approx f(0,0,0) + \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial u_1}u_1 + \frac{\partial f}{\partial u_2}u_2 \end{aligned}$$

where the partial derivatives are evaluated at the operating point, namely at $(x, u_1, u_2) = (0,0,0)$. We have $f(0,0,0) = 0$ and:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left. \frac{-1}{10\sqrt{(\frac{4}{5} + u_2) + \frac{1}{5}(1+x)}} \right|_{0,0,0} = -\frac{1}{10} \\ \frac{\partial f}{\partial u_1} &= 1 \\ \frac{\partial f}{\partial u_2} &= \left. \frac{-1}{2\sqrt{(\frac{4}{5} + u_2) + \frac{1}{5}(1+x)}} \right|_{0,0,0} = -\frac{1}{2} \end{aligned}$$

Hence the linearised state equation is:

$$\dot{x} = -\frac{1}{10}x + u_1 - \frac{1}{2}u_2$$

which is in the standard form with

$$A = -\frac{1}{10} \quad \text{and} \quad B = \left[1, -\frac{1}{2}\right]$$

We have $M_o = 1 + y_1 = \sqrt{(\frac{4}{5} + u_2) + \frac{1}{5}(1+x)} \triangleq g(x, u_2)$ so the output equation is

$$\begin{aligned}
\begin{bmatrix} 1 + y_1 \\ 1 + x \end{bmatrix} &\approx \begin{bmatrix} g(0,0) + \frac{\partial g}{\partial x}x + \frac{\partial g}{\partial u_2}u_2 \\ 1 + x \end{bmatrix} \\
&= \begin{bmatrix} 1 + \frac{1}{10}x + \frac{1}{2}u_2 \\ 1 + x \end{bmatrix}
\end{aligned}$$

and hence the linearised output equation is

$$\begin{bmatrix} y_1 \\ x \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

which is in the standard form with

$$C = \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

(d) From part 1a we have

$$\begin{aligned}
G(s) &= C(sI - A)^{-1}B + D \\
&= \frac{\begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix} [1, -\frac{1}{2}]}{s + \frac{1}{10}} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \\
&= \frac{\begin{bmatrix} \frac{1}{10} & -\frac{1}{20} \\ 1 & -\frac{1}{2} \end{bmatrix}}{s + \frac{1}{10}} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \\
&= \frac{\begin{bmatrix} \frac{1}{10} & \frac{s}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}}{s + \frac{1}{10}}
\end{aligned}$$

2. (a) Let k_0 be the gain required to put two closed-loop poles at -4. Then

$$1 + k_0 G(-4) = 0$$

hence

$$k_0 = \frac{1}{|G(-4)|} = \frac{|-4+2| \times |(-4)^2 + 12(-4) + 40|}{1} = 2 \times 8 = 16$$

(or, using the geometry shown in Fig.2: $k_0 = 2 \times 2\sqrt{2} \times 2\sqrt{2} = 16$.)
With this value of gain the closed-loop characteristic equation is

$$\begin{aligned} 1 + k_0 G(s) &= 0 \\ 1 + k_0 \frac{1}{(s+2)(s^2+12s+40)} &= 0 \end{aligned}$$

which has the same roots as

$$\begin{aligned} (s+2)(s^2+12s+40) + k_0 &= 0 \\ s^3 + 14s^2 + 64s + 80 + k_0 &= 0 \\ s^3 + 14s^2 + 64s + 80 + 16 &= 0 \\ s^3 + 14s^2 + 64s + 96 &= 0 \end{aligned}$$

But we know that this has 2 roots at -4, so letting the third root be p we have

$$s^3 + 14s^2 + 64s + 96 \equiv (s+4)^2(s-p)$$

Hence (comparing the constant terms) $96 = 4^2 \times (-p)$ so $\boxed{p = -6}$.

(b) Let k_1 be the proportional gain at which the system just loses stability. Then the gain margin is k_1/k_0. To find k_1, note from Fig.2 that the root loci cross the imaginary axis at $\pm 8j$. So we find the gain k_1 as before:

$$\begin{aligned} k_1 &= \frac{1}{|G(8j)|} = \frac{|8j+2| \times |(8j)^2 + 12(8j) + 40|}{1} \\ &= |8j+2| \times |8j - (-6+2j)| \times |8j - (-6-2j)| \\ &= \sqrt{64+4} \times \sqrt{36+36} \times \sqrt{36+100} \\ &= 2\sqrt{17} \times 6\sqrt{2} \times 2\sqrt{2}\sqrt{17} \\ &= 816 \end{aligned}$$

Hence the gain margin is $816/16 = \boxed{51}$

(c) Note that

$$1 + \frac{3}{s} = \frac{s+3}{s}$$

so we need the root-locus diagram of

$$\frac{s+3}{s(s+2)(s^2+12s+40)}$$

There is one zero at -3, and four poles at 0, -2 and $-6 \pm 2j$.
Hence there are $4 - 1 = 3$ asymptotes, which will make angles $\pi/3$, π and $5\pi/3$ with the positive real axis.
These asymptotes intersect at the 'centre of gravity' of the poles and zeros:

$$\frac{[0 - 2 + (-6+2j) + (-6-2j)] - [-3]}{4-1} = \frac{-2-12+3}{3} = -\frac{11}{3}$$

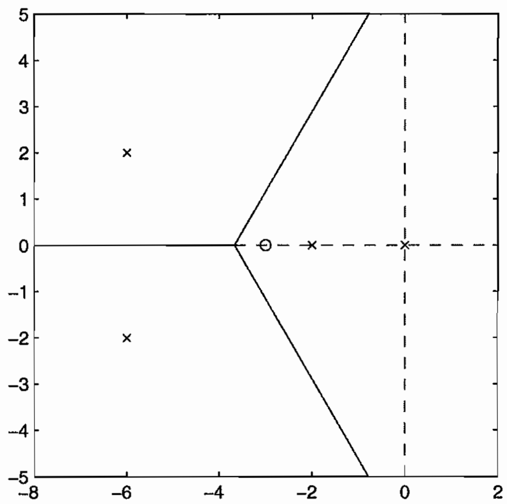

Figure 1:

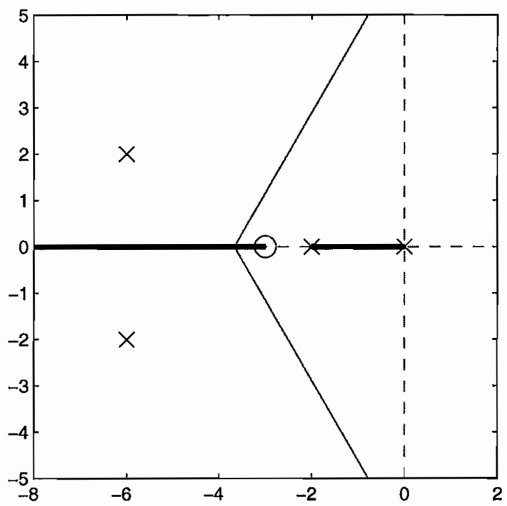

Figure 2:

So the pole-zero plot, with asymptotes, looks like Figure 1.

Now adding the portions of the real axis which are to the left of an odd number of poles and zeros gives the partial root-locus diagram shown in Figure 2.

There must be breakaway points between 0 and -2, and to the left of -3. Since only a 'sketch' is asked for, a guess can be made at the locations of these, and at the shape of the remaining loci, giving Figure 3. *Note: This is not the only possible form. Credit will be given for other plausible forms — there is only one.*

(d) The benefit of using a PI controller would be that integral action would allow a demanded speed to be followed without error (in the steady-state) despite variations in the material being machined (eg

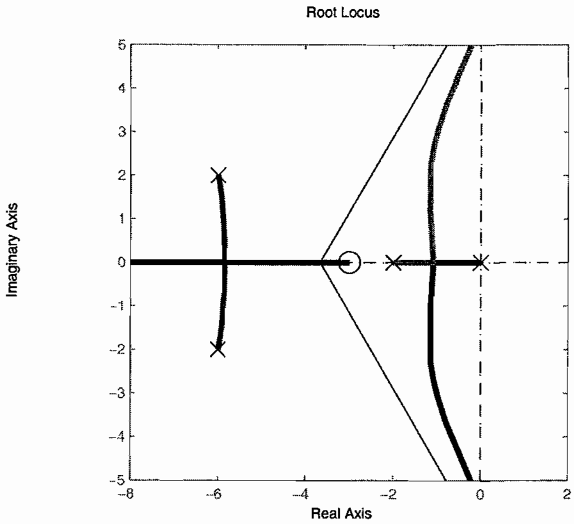

Figure 3:

variations of hardness, imperfections, etc).

3. (a) Standard test for controllability: if n is the dimension of the state vector, then the system $\dot{x} = Ax+Bu$ is controllable if and only if

$$\text{rank}[B, AB, A^2B, \ldots, A^{n-1}B] = n \tag{1}$$

(b) The effect of an input trajectory $u(t)$, $0 \le t < t_1$, on the state is given by

$$\begin{aligned} x(t_1) &= \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \\ &= \int_0^{t_1} [I + A(t_1-\tau) + A^2\frac{(t_1-\tau)^2}{2!} + \cdots]Bu(\tau)d\tau \\ &= \int_0^{t_1} [B + AB(t_1-\tau) + A^2B\frac{(t_1-\tau)^2}{2!} + \cdots]u(\tau)d\tau \end{aligned}$$

If the rank test in part 3a is not satisfied, so the rank is smaller than n, then the columns of $[B, AB, A^2B, \ldots, A^{n-1}B]$ do not span the whole state space. But, as a consequence of the Cayley-Hamilton theorem, the columns of A^kB, where $k \ge n$, are in the span of the columns of $[B, AB, A^2B, \ldots, A^{n-}$
Hence $x(t_1)$ is confined to lie in the column span of this matrix, and hence cannot be driven in other directions in the state space, no matter what input trajectory is chosen.

(c) Let the state vector be $x = [\theta, \dot{\theta}, z, \dot{z}]^T$. Then

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{z} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ z \\ \dot{z} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \\ 0 \\ 0 \end{bmatrix} u \tag{2}$$

(d) Applying the test (1) to equation (2) with $n = 4$:

$$\text{rank}[B, AB, A^2B, A^3B] = \text{rank} \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\gamma \\ 0 & 0 & \alpha\gamma & 0 \end{bmatrix} = 4$$

since $\alpha > 0$ and $\gamma > 0$.Hence the ball and beam apparatus is controllable.
(*Note:* Evaluate A^2B as $A(AB)$ etc to minimise amount of computation required.)

(e) The torque is given by $u = -k^Tx$, where $k = [0, k_1, k_3, k_2]^T$. Hence

$$\dot{x} = Ax + B(-k^Tx) = (A - Bk^T)x$$

so closed-loop stability is determined by the eigenvalues of $(A - Bk^T)$. Now

$$\begin{aligned} A - Bk^T &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha k_1 & \alpha k_3 & \alpha k_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\alpha k_1 & -\alpha k_3 & -\alpha k_2 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence

$$\lambda I - (A - Bk^T) = \begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda + \alpha k_1 & \alpha k_3 & \alpha k_2 \\ 0 & 0 & \lambda & -1 \\ -\gamma & 0 & 0 & \lambda \end{bmatrix}$$

so that (expanding the determinant by the first row):

$$\begin{aligned} \det[\lambda I - (A - Bk^T)] &= \lambda[\lambda^2(\lambda + \alpha k_1)] + \det \begin{bmatrix} 0 & \alpha k_3 & \alpha k_2 \\ 0 & \lambda & -1 \\ -\gamma & 0 & \lambda \end{bmatrix} \\ &= \lambda^3(\lambda + \alpha k_1) - \gamma(-\alpha k_3 - \lambda k_2) \\ &= \lambda^4 + \alpha k_1 \lambda^3 + \alpha\gamma k_2 \lambda + \alpha\gamma k_3 \end{aligned}$$

In this polynomial the coefficient of λ^2 is zero. Hence by the Routh-Hurwitz criterion, the roots cannot all have negative real parts, so it is impossible to achieve asymptotic stability.

Alternative without using state-space method: Using transfer functions, we have

$$\frac{\dot{\theta}}{u} = \frac{\alpha}{s}, \quad \frac{\dot{z}}{\theta} = \frac{\gamma}{s}, \quad \frac{z}{\theta} = \frac{\gamma}{s^2}$$

Hence with the proposed feedback we have

$$u = \left[-k_1 \frac{\alpha}{s} - k_2 \frac{\alpha\gamma}{s^3} - k_3 \frac{\alpha\gamma}{s^4}\right] u$$

which gives (multiplying through by s^4) the closed-loop characteristic equation

$$s^4 + k_1 \alpha s^3 + k_2 \alpha\gamma s + k_3 \alpha\gamma = 0$$

and now the same argument applies as above.

4. (a) The observer block diagram is standard.

The *purpose* of a state observer is to form estimates of all the state variables from measurements of inputs and outputs of a dynamic system.

The *operation* of a state observer is best described by reference to a standard block-diagram. The observer (with state $\hat{x}$) is a copy of the plant (with state x). The equations of the combined system are given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \end{aligned}$$

where $y = Cx$ and $\hat{y} = C\hat{x}$ (assuming $D = 0$ for simplicity). Defining the state estimation error $e = x - \hat{x}$ gives

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= (A - LC)e \end{aligned}$$

so the error decays to zero at a rate dependent on the eigenvalues of $A - LC$ (also known as the 'observer poles'), if these all have negative real parts.

(b) The first equation is second-order, and the second one is first-order, so we expect to need 3 state variables. Let the state vector be $x = [\dot{\theta}, \theta, v]^T$. Then the given equations can be written as

$$\dot{x} = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} f = Ax + Bf \tag{3}$$

which is in the standard state-space form.

(c) Suppose that only θ is measured. Then $\theta = C_1 x$ requires

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{4}$$

Since the state dimension is $n = 3$, the system is observable if

$$\text{rank} \begin{bmatrix} C_1 \\ C_1 A \\ C_1 A^2 \end{bmatrix} = 3 \tag{5}$$

We have

$$\begin{bmatrix} C_1 \\ C_1 A \\ C_1 A^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{6}$$

which clearly has rank = 3 by inspection. (3 linearly independent rows or columns. Or compute determinant and see that it is not 0.) So the system is observable.

Now suppose that only v is measured. Then $v = C_2 x$ requires

$$C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{7}$$

We have

$$\begin{bmatrix} C_2 \\ C_2 A \\ C_2 A^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \tag{8}$$

which again has rank = 3 by inspection. So again the system is observable.

(d) If θ is measured but v is not, then we have C_1 as the output matrix. The entries of the observer gain matrix L need to be designed. Since there is one measured output and 3 states, the dimensions of L are 3×1. The observer poles are the eigenvalues of the matrix $A - LC$ (see above). We have

$$
\begin{aligned}
A - LC &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1-\ell_1 & 1 \\ 1 & -\ell_2 & 0 \\ 0 & 1-\ell_3 & -1 \end{bmatrix}
\end{aligned}
$$

Now

$$
\begin{aligned}
\det[\lambda I - (A - LC)] &= \det \begin{bmatrix} \lambda & \ell_1 - 1 & -1 \\ -1 & \lambda + \ell_2 & 0 \\ 0 & \ell_3 - 1 & \lambda + 1 \end{bmatrix} \\
&= \lambda[(\lambda + \ell_2)(\lambda + 1) - 0] + [(\ell_1 - 1)(\lambda + 1) + (\ell_3 - 1)] \\
&= \lambda^3 + (1 + \ell_2)\lambda^2 + (\ell_2 + \ell_1 - 1)\lambda + (\ell_1 + \ell_3 - 2)
\end{aligned}
$$

But we want this to be the same as

$$
(\lambda + 1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1
$$

so we must have

$$
\begin{aligned}
1 + \ell_2 = 3 \quad &\Rightarrow \quad \ell_2 = 2 \\
\ell_2 + \ell_1 - 1 = 3 \quad &\Rightarrow \quad \ell_1 = 2 \\
\ell_1 + \ell_3 - 2 = 1 \quad &\Rightarrow \quad \ell_3 = 1
\end{aligned}
$$

(e) Possible reasons for using both θ and v measurements for estimating x:

- The model is always approximate, so the state estimation results are always approximate in practice. Therefore having a direct measurement of another state variable will improve the quality of estimation of that variable, as well as that of other states.
- Measurements are often noisy. Having more measurements gives the potential for improving the signal-to-noise ratio. In particular, if noise statistics are known, the observer can be designed so as to optimally combine the measurements to get the best quality state estimates (Kalman filter).
- Estimation of sensor bias may be possible if more than one sensor is used.
- Measurements may be distorted by sensor characteristics. In particular, angle and velocity sensors may have different performance in different frequency ranges. Using both measurements may allow accurate coverage of a wider frequency range.
- Even though the system may be observable using only one measurement, it may be very weakly observable (observability matrix Q nearly singular). In this case a second measurement may improve the quality of the state estimate.
- A sensor may fail. Having two different ones available may allow the system to continue working after one fails.

3F2 Systems and Control: 2006 Numerical answers

1. (a) —
   (b) —
   (c)

$$A = -\frac{1}{10}, \qquad B = [1, -\tfrac{1}{2}]$$

$$C = \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

   (d) —

2. (a) Gain: 16. Third pole: -6.
   (b) Exact value is 51.
   (c) —
   (d) —

3. (a) —
   (b) —
   (c) If $x = [\theta, \dot{\theta}, z, \dot{z}]^T$ then

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \alpha \\ 0 \\ 0 \end{bmatrix}$$

   (d) —
   (e) —

4. (a) —
   (b) If $x = [\dot{\theta}, \theta, v]^T$ then

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

   (c) —
   (d)

$$L = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

   (e) —