PART IIA 2006
3F3: Signal and pattern processing
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Datasheet: ***None***

ENGINEERING TRIPOS PART IIA

Saturday 13 May 2006 9 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

Worked Solutions

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The Discrete Fourier Transform (DFT) for a data sequence $\{x_n\}$ of length N is defined as

$$X_p = \sum_{n=0}^{N-1} x_n \, e^{-j\frac{2\pi}{N}np}, \quad p = 0, 1, \ldots, N-1$$

(a) Describe the radix-2 Fast Fourier Transform (FFT) algorithm for implementation of the above DFT when N is a power of 2. Your description should include: an expression for the original length N DFT in terms of two length $N/2$ DFTs; the number of 'stages' in the FFT algorithm; bit reversal operations; in-place computation; and the butterfly structure.

What is the computational load (in complex multiplies and additions) for the FFT? [40%]

Solution:

First, take the basic DFT equation:

$$X_p = \sum_{n=0}^{N-1} x_n \, e^{-j\frac{2\pi}{N}np}$$

Now, split the summation into two parts: one for even n and one for odd n:

$$\begin{aligned}
X_p &= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} \, e^{-j\frac{2\pi}{N}(2n)p} + \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-j\frac{2\pi}{N}(2n+1)p} \quad (*) \\
&= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} e^{-j\frac{2\pi}{(N/2)}np} + e^{-j\frac{2\pi}{N}p} \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-j\frac{2\pi}{(N/2)}np} \\
&= A_p + W^p B_p
\end{aligned}$$

where

$$\begin{aligned}
A_p &= \sum_{n=0}^{\frac{N}{2}-1} x_{2n} e^{-j\frac{2\pi}{(N/2)}np} \\
B_p &= \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1} e^{-j\frac{2\pi}{(N/2)}np} \\
W &= e^{-j\frac{2\pi}{N}}
\end{aligned}$$

and it can be seen that A_p and B_p are each N/2 length DFTs

To see how this simplifies, look at the original DFT in (*) above, but evaluated at frequencies $p + N/2$.

It turns out that we can write:

$$X_{p+N/2} = \sum_{n=0}^{\frac{N}{2}-1} x_{2n}\, e^{-j\frac{2\pi}{(N/2)}np} - e^{-j\frac{2\pi}{N}p} \sum_{n=0}^{\frac{N}{2}-1} x_{2n+1}\, e^{-j\frac{2\pi}{(N/2)}np}$$

$$= A_p - W^p B_p$$

with A_p W^p and B_p defined as before

Now, compare the equations for $X_{p+N/2}$ with that for X_p :

$$X_p = A_p + W^p B_p, \quad X_{p+N/2} = A_p - W^p B_p$$

[This is more detailed than required - a qualitatative description is fine]

This defines the FFT butterfly structure.

Each N/2 length DFT is then split again, into N/4 length DFTs until we have decomposed to length 1 DFTs (since N is a power of 2).

There are $M = \log_2(N)$ stages stages in this decomposition, each using N / 2 butterflies.

Each stage can be computed 'in place' overwriting the same memory location

In order to get the correct output spectrum, the input data are reordered by 'bit-reversal' of the binary expansion of their sample number.

Each butterfly takes one complex multiply ($W \times x$) plus two complex additions. Hence, in total we have $N/2 \log_2(N)$ complex multiplies and $N \log_2(N)$ complex additions.

(b) Let $\{x_n\}$, $n = 0, 1, ..., N-1$, be a sequence of N data points with DFT $\{X_p\}$, where N is once again a power of 2. The sequence is known to satisfy

$$x_{n+N/2} = -x_n$$

for $n = 0, 1, \ldots, \frac{N}{2} - 1$.

Show that the DFT $\{X_p\}$ is zero for all *even* values of the frequency index p , i.e. show that $X_{2m} = 0$, for $m = 0, 1, ..., N/2 - 1$. [30%]

Solution:

$$
\begin{aligned}
X_p &= \sum_{n=0}^{N-1} x_n \, e^{-j\frac{2\pi}{N}np} \\
&= \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}np} + \sum_{n=N/2}^{N-1} x_n \, e^{-j\frac{2\pi}{N}np} \\
&= \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}np} + \sum_{n=0}^{N/2-1} x_{n+N/2} \, e^{-j\frac{2\pi}{N}(n+N/2)p} \\
&= \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}np} + \sum_{n=0}^{N/2-1} -x_n \, e^{-j\pi p} e^{-j\frac{2\pi}{N}np} \\
&= 0, \quad \text{(p even)} \\
&= 2 \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}np}, \quad \text{(p odd)}
\end{aligned}
$$

[since $\exp(-j\pi p) = -1^p$].

(c) Consider the modified sequence of $N/2$ data points

$$y_n = x_n \exp(-2jn\pi/N), \;\; n = 0, 1, ..., N/2 - 1$$

where $\{x_n\}$ is a sequence with properties as in part (b).

Write down an expression for the DFT $\{Y_p\}$ of the length $N/2$ sequence $\{y_n\}$. Starting from this expression, show that the non-zero values of the DFT $\{X_p\}$ in part (b) may be calculated as

$$X_{2m+1} = 2Y_m, \;\; m = 0, 1, ... N/2 - 1$$

[30%]

Solution:

$$\begin{aligned}
Y_p &= \sum_{n=0}^{N/2-1} y_n \, e^{-j\frac{2\pi}{N/2}np}, \quad p = 0, 1, ..., N/2-1 \\
&= \sum_{n=0}^{N/2-1} x_n \exp(-2jn\pi/N) \, e^{-j\frac{2\pi}{N/2}np}, \quad p = 0, 1, ..., N/2-1 \\
&= \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}(2p+1)n}
\end{aligned}$$

But from part b) in the case of 'p odd', we have that

$$X_{2p+1} = 2 \sum_{n=0}^{N/2-1} x_n \, e^{-j\frac{2\pi}{N}(2p+1)n}$$

Hence $2Y_p = X_{2p+1}$ as required.

2 (a) Explain how the bilinear transform

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

can be used for the design of a digital filter from an analogue prototype filter. Your explanation should include the constraints on the type of filter that can usefully be designed, any distortions that are introduced, and procedures for transforming between one type of analogue prototype filter and another (low-pass to high-pass, for example). [40%]

Solution:

The steps of the bilinear transform method are as follows:

1. Warp the digital critical (e.g. bandedge or "corner") frequencies ω_i , in other words compute the corresponding analogue critical frequencies $\omega_i = tan(\Omega_i/2)$.

2. Design an analogue filter which satisfies the resulting filter response specification.

3. Apply the bilinear transform to the s-domain transfer function of the analogue filter to generate the required z-domain transfer function.

Can design IIR filters with this method from analogue prototype. Distortions are introduced in the non-linear frequency warping - will distort phases even in sections with constant amplitude response. To get one type of filter from another, use transformation functions in the analogue domain (also possible in the digital domain, but not covered in this course):

Lowpass to Lowpass: Set $s' = s/\omega_c$ to give lowpass with cutoff at ω_c

Lowpass to Highpass: $s' = \frac{s^2+\omega_l\omega_u}{s(\omega_u-\omega_l)}$ to give bandpass with lower cutoff at ω_l , upper cutoff at w_u

Lowpass to Bandpass: $s' = \frac{s^2+\omega_l\omega_u}{s(\omega_u-\omega_l)}$ to give bandpass with lower cutoff at ω_l , upper cutoff at w_u

Lowpass to Bandstop: $s' = \frac{s(\omega_u-\omega_l)}{s^2+\omega_l\omega_u}$ to give bandstop with lower cutoff at ω_l , upper cutoff at w_u

[One or two of the above examples would suffice]

(b) Show that the bilinear transform in (a) maps the unit circle in the z-plane onto the entire imaginary axis of the s-plane, according to a frequency warping function

$$\omega = \tan(\Omega/2)$$

where ω rad s^{-1} is the frequency in the analogue domain and Ω rad/sample is the normalised digital frequency. [20%]

Solution:

Let $\psi(z)$ be the bilinear transform. Then set $z = \exp(j\omega)$. Now:

$$\begin{aligned}
\psi(\exp(j\Omega)) &= \frac{1-\exp(-j\Omega)}{1+\exp(-j\Omega)} \\
&= \frac{\exp(-j\Omega/2)(\exp(j\Omega/2)-\exp(-j\Omega/2))}{\exp(-j\Omega/2)(\exp(+j\Omega/2)+\exp(-j\Omega/2))} \\
&= \frac{j\sin(\Omega/2)}{\cos(\Omega/2)} \\
&= j\tan(\Omega/2)
\end{aligned}$$

which is purely imaginary, with the required frequency warp.

(c) In a high frequency audio perception experiment it is required to extract frequency components in a signal which lie *above* 20kHz, in a digital system having sampling rate 96 kHz. A low-pass Butterworth prototype filter is available having cut-off frequency 1 rad s^{-1} and transfer function

$$H(s) = \frac{1}{(s^2+\sqrt{2}s+1)}.$$

Use this analogue prototype and the bilinear transform to obtain the transfer function of a possible digital filter for the experiment. Without additional calculations, comment on whether you would expect this filter to perform adequately in extracting high frequency components from the audio. [40%]

Solution:

Cut-off frequency of analogue filter use:

$$\omega_c = \tan(\Omega_c/2) = \tan((20 \times 2\pi/96)/2) = 0.767$$

For a high-pass filter, use transformation

$$s' = \omega_c/s$$

Hence analogue prototype becomes:

$$H(s') = \frac{1}{(\omega_c/s')^2 + \sqrt{2} * (\omega_c/s') + 1))} = \frac{s'^2}{\omega_c^2 + \sqrt{2} * \omega_c * s' + s'^2}$$

Now apply bilinear transform:

$$s' \rightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$\begin{aligned} H(z) &= \frac{\frac{1-z^{-1}}{1+z^{-1}}^2}{\omega_c^2 + \sqrt{2} * \omega_c * \frac{1-z^{-1}}{1+z^{-1}} + \frac{1-z^{-1}}{1+z^{-1}}^2} \\ &= \frac{\frac{1-z^{-1}}{1+z^{-1}}^2}{\omega_c^2 + \sqrt{2} * \omega_c * \frac{1-z^{-1}}{1+z^{-1}} + \frac{1-z^{-1}}{1+z^{-1}}^2} \\ &= \frac{1/2.67(1 - 2z^{-1} + z^{-2})}{1 + 0.30z^{-1} + 0.19z^{-2}} \end{aligned}$$

This is a very low order filter for a high quality audio experiment - would expect significant audible residual in the low frequency range owing to the shallow cut-off expected from such a filter.

[Many candidates thought that aliasing would be the problem - however, this is another issue altogether].

3 (a) For a discrete-time random process, define the terms autocorrelation function, wide sense stationarity (WSS), power spectrum and mean ergodicity. [20%]

Solution:

Autocorrelation function:

$R_{XX}[k_1, k_2] = E[X_{k_1} X_{k_2}]$

A random process is *wide-sense stationary (WSS)* if:

(i) $\mu_n = E[X_n] = \mu$, (mean is constant)

(ii) $r_{XX}[n, m] = r_{XX}[m-n]$, (autocorrelation function depends only upon the difference between n and m).

(iii) The variance of the process is finite:

$$E[(X_n - \mu)^2] < \infty$$

For a wide-sense stationary random process $\{X_n\}$, the power spectrum is defined as the discrete-time Fourier transform (DTFT) of the discrete autocorrelation function:

$$\boxed{S_X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} r_{XX}[m]\, e^{-jm\Omega}} \quad (1)$$

Power spectrum for a random process

where $\Omega = \omega T$ is used for convenience.

For a **Mean Ergodic** random process we can estimate the expectation by performing time-averaging on a single sample function

$$\mu = E[X_n] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n \text{ (Mean ergodic)}$$

(b) When a WSS random process is input to a linear, stable, time-invariant system, derive an expression for the autocorrelation function of the ourput random process, in terms of the input autocorrelation function and the impulse response of the linear system. Hence show that the output power spectrum can be expressed as

$$S_Y(e^{j\theta}) = |H(e^{j\theta})|^2 S_X(e^{j\theta})$$

where $S_X(e^{j\theta})$ is the input power spectrum and $H(e^{j\theta})$ is the frequency response of the linear system [40%]

Solution:

We can express the output correlation functions and power spectra in terms of the input statistics and the LTI system:

$$r_{XY}[k] = E[X_n Y_{n+k}] = \sum_{l=-\infty}^{\infty} h_l \, r_{XX}[k-l] = h_k * r_{XX}[k] \qquad (2)$$

Cross-correlation function at the output of a LTI system

$$r_{YY}[l] = E[Y_n Y_{n+l}]$$
$$\sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} h_k \, h_i \, r_{XX}[l+i-k] = h_l * h_{-l} * r_{XX}[l] \qquad (3)$$

Autocorrelation function at the output of a LTI system

Taking DTFT of both sides of (3):

$$\boxed{\mathcal{S}_Y(e^{j\omega T}) = |H(e^{j\omega T})|^2 \mathcal{S}_X(e^{j\omega T})} \qquad (4)$$

Power spectrum at the output of a LTI system

(c) A zero-mean white noise process $\{w_n\}$ is filtered through the system

$$y_n = 0.9\, y_{n-1} - 0.81\, y_{n-2} + w_n$$

Determine the poles and zeros of the system and determine its frequency response. Determine also the power spectrum of y_n and sketch it, assuming that $E[w_n]^2 = 1$. [40%]

Solution:

Transfer function is:

$$H(z) = \frac{1}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

Hence poles are roots of denominator equation:

$$r = 0.9, \ \ \theta = \pm \cos^{-1}(0.5) = 60^0$$

or

$$z = 0.45 \pm 0.78i$$

No zeros (strictly 2 at origin, but these correspond to a pure delay of 2 samples (z^{-2})).

Frequency response is obtained as $H(\exp(j\Omega))$:

$$H(\exp(j\Omega)) = \frac{1}{1 - 0.9\exp(-j\Omega) + 0.81\exp(-2j\Omega)}$$

Hence power spectrum is obtained as:

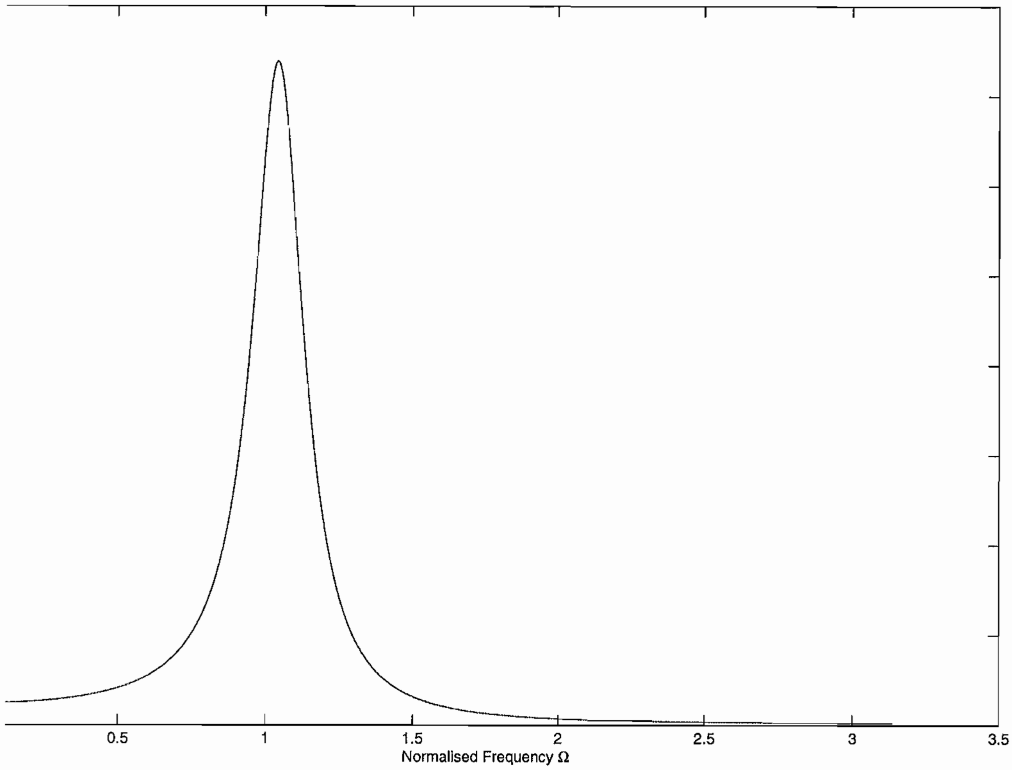
$$\begin{aligned} \mathcal{S}_Y(e^{j\omega T}) &= |H(e^{j\omega T})|^2 \mathcal{S}_X(e^{j\omega T}) \\ &= |\frac{1}{1 - 0.9\exp(-j\Omega) + 0.81\exp(-2j\Omega)}|^2 \times 1 \end{aligned}$$

Note that $\mathcal{S}_X(e^{j\omega T}) = 1$ since the input is white noise having unit impulse autocorrelation function, whose DTFT equals unity.

Sketch this function by consideration of the frequency response, i.e. having poles at

$$z = 0.45 \pm 0.78i$$

, and hence quite sharp peaks around $\pm 60^0$. Sketching from 0 to π gives:


0.5
1
1.5
2
2.5
3
3.5
Normalised Frequency Ω

4 Consider a study measuring the correlation between alcohol consumption and examination marks for Cambridge Undergraduates. The variable x measures units of alcohol consumed daily, while y measures marks out of 100. The following data are collected from 4 subjects in the study, $D = \{(0,80),(1,60),(3,60),(4,40)\}$, where each data point is an (x,y) pair.

Assume a linear regression model

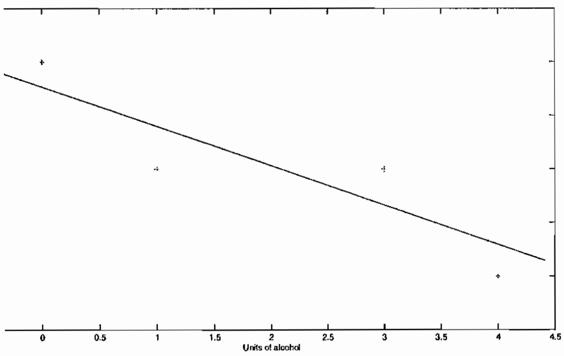
$$y_i = ax_i + b + \epsilon_i$$

where a and b are the slope and intercept of the regression line, respectively, and ϵ is zero mean Gaussian noise with variance σ^2. That is, $p(y_i|x_i,a,b,\sigma)$ is a Gaussian with mean $ax_i + b$ and variance σ^2, where $i \in \{1,\ldots,4\}$ indexes the subjects in the study.

Assume all the data points were generated independently from this model.

(a) Sketch the position of the points on a diagram and draw the approximate linear regression line. [15%]

Answer:



(b) Write down the log likelihood as a function of the model parameters a, b, and σ. [15%]

Answer:

$$L = -2\log(2\pi) - 4\log\sigma - \frac{1}{2\sigma^2}\left[(80-b)^2 + (60-a-b)^2 + (60-3a-b)^2 + (40-4a-b)^2\right]$$

(c) Assume $\sigma = 10$. Solve for the maximum likelihood settings of the parameters a and b. [40%]

Answer: Taking derivatives wrt a , ignoring proportionality constant $\frac{1}{2\sigma^2}$:

$$\begin{aligned}\frac{dL}{da} &\propto -2(60-a-b)-6(60-3a-b)-8(40-4a-b)=0 \\ &= -120+2a+2b-360+18a+6b-320+32a+8b=0 \\ 52a+16b &= 800\end{aligned}$$

Taking derivative wrt b :

$$\begin{aligned}\frac{dL}{db} &\propto -2(80-b)-2(60-a-b)-2(60-3a-b)-2(40-4a-b)=0 \\ & \quad 80-b+60-a-b+60-3a-b+40-4a-b=0 \\ 4b+8a &= 240 \\ b &= 60-2a\end{aligned}$$

Solution: $a=-8$ and $b=76$.

(d) Given the values of a and b found in part (c), do you think that $\sigma=10$ is a reasonable estimate, an overestimate, or an underestimate of σ ? Justify your answer. [15%]

Answer: The optimal value of σ is in fact $\sqrt{40}$ which can be computing by maximizing the likelihood.

$$\begin{aligned}\frac{dL}{d\sigma} &= -\frac{4}{\sigma}+\frac{1}{\sigma^3}[4^2+8^2+8^2+4^2]=0 \\ \sigma^2 &= \frac{1}{4}[16+64+64+16]=40\end{aligned}$$

So $\sigma=10$ is an overestimate.

(e) Given that examination marks range between 0 and 100, do you think the assumptions made in this model are reasonable? Justify your answer. [15%]

Answer: The assumptions are not reasonable if we extrapolate far from the data. Gaussianity implies that values of the marks could range outside [0, 100]. Linearity implies that extrapolating one could get negative marks if one drank too much. A nonlinear model with non-Gaussian noise would be more reasonable.

END OF PAPER