ENGINEERING TRIPOS PART IIA

Tuesday 9 May 2006 9 to 10.30

Module 3C5

DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A symmetrical spinning top of mass m has principal moments of intertia AAC about axes passing through the fixed point O in contact with a horizontal table as shown in Fig. 1. The distance from O to the centre of mass of the top is a. The top is spinning at steady rate ω about its axis of symmetry.

(a) By using the Gyroscope Equations or otherwise find a general expression for the rate of steady precession when the axis is inclined at an angle θ to the vertical. [40%]

(b) Show that the precession rate approximates to $\dfrac{mga}{C\omega}$ for fast spin. [10%]

(c) A top will not stand up unless it is spinning fast enough. Use your answer from (a) to determine the rate of spin below which the top will not stand up. [40%]

(d) Discuss briefly whether your answer is truly valid for $\theta = 0$. [10%]

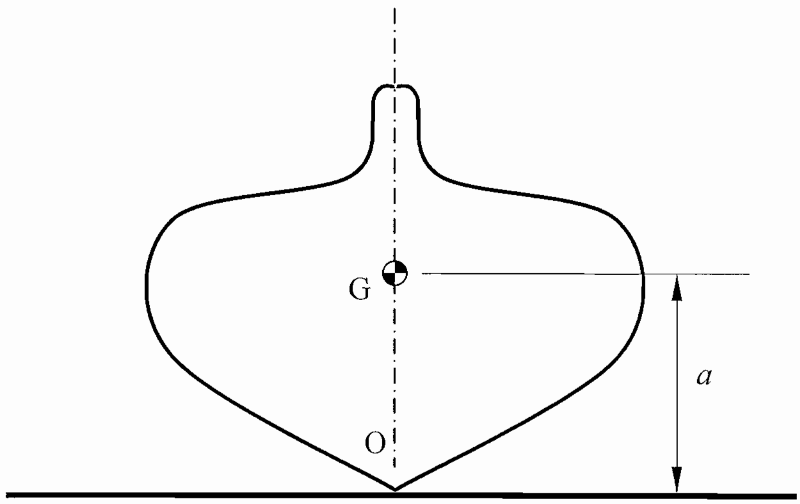

Fig. 1

2 An assembly comprises a thin uniform square plate OABC of mass $6m$ and side $2a$ and a thin uniform rod OD of mass $6m$ and length $2a$. The rod is attached to the plate at O with OD perpendicular to the plane of the plate as shown in Fig. 2. The centre of the plate is at E and the centre of the rod is at F. A Cartesian reference frame (x, y, z) has its origin at O as shown.

(a) Locate the (x, y, z) coordinates of G, the centre of mass of the assembly. [10%]

(b) Find the inertia matrix in the (x, y, z) frame at O and verify that the moments of inertia are each equal to $16ma^2$ and that the only non-zero products of inertia are each equal to $6ma^2$. [30%]

(c) Find the inertia matrix at G in a frame (x', y', z') parallel to (x, y, z) and verify that the products of inertia are each equal to $\pm 3ma^2$. [40%]

(d) Show that EF is a principal axis and find the principal moment of inertia along EF. [20%]

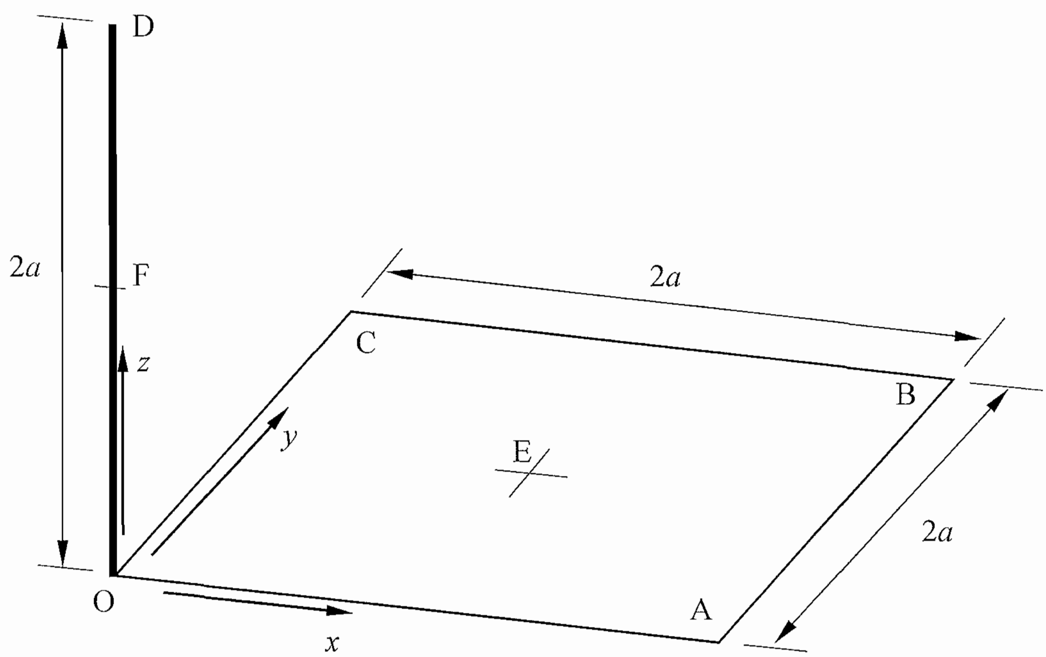

Fig. 2

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3 A flat circular table whose centre is at O is inclined at a small angle α to the horizontal as shown in Fig. 3. It is rotating about its axis of symmetry at steady rate $\Omega\boldsymbol{k}$ where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is a *space-fixed* reference frame with $\boldsymbol{j}$ aligned with the horizontal. A solid sphere S of mass m and radius a is observed to be rolling without slip on the surface of the table. The sphere is moving steadily with constant velocity $v\boldsymbol{j}$ along the horizontal line AOB (which is fixed in space) as shown in the figure.

(a) Draw a free-body diagram of the ball and hence show that:

(i) the table exerts a constant force on the ball vertically upwards; [10%]

(ii) the couples Q_1 and Q_3 (as defined for use with Euler's equations in the Datasheet) are zero; [10%]

(iii) the magnitude of the angular acceleration of the ball is $\dfrac{5g\sin\alpha}{2a}$. [20%]

(b) Is the motion of the ball governed by gyroscopic effects? Give reasons for your answer. [10%]

(c) Use a no-slip condition to find an expression for the velocity v of the ball. [50%]

(cont.

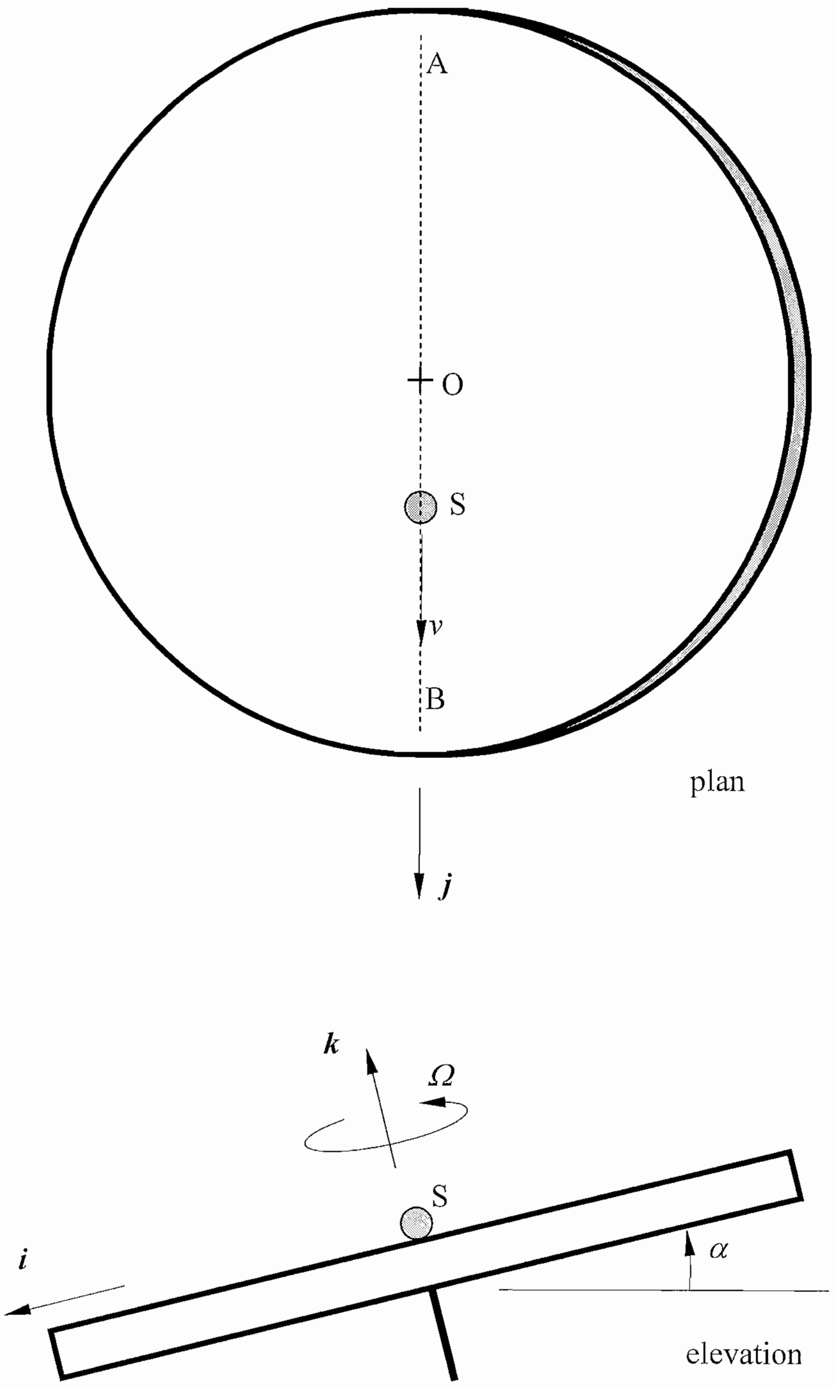

Fig. 3

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4 Figure 4 shows two objects each of mass m that move in free space under the action of their mutual gravitational attraction. For planar motion there are four degrees of freedom which are taken to be the displacements x and y of the centre of mass C, the distance r between each mass and the centre of mass and the angle θ shown in the figure. The gravitational potential energy is $V = -\dfrac{Gm^2}{2r}$ where G is the universal gravitational constant.

(a) By using Lagrange's equations derive the equations of planar motion using the four generalized coordinates x, y, r and θ. [50%]

(b) Show that $\dot{x}$, $\dot{y}$ and $r^2\dot{\theta}$ are all constant during the motion, and explain these results in physical terms. [20%]

(c) One possible solution to the equations of motion is steady circular motion of each mass around the centre of gravity. Find in terms of G, m and r an expression for the period of this motion. [30%]

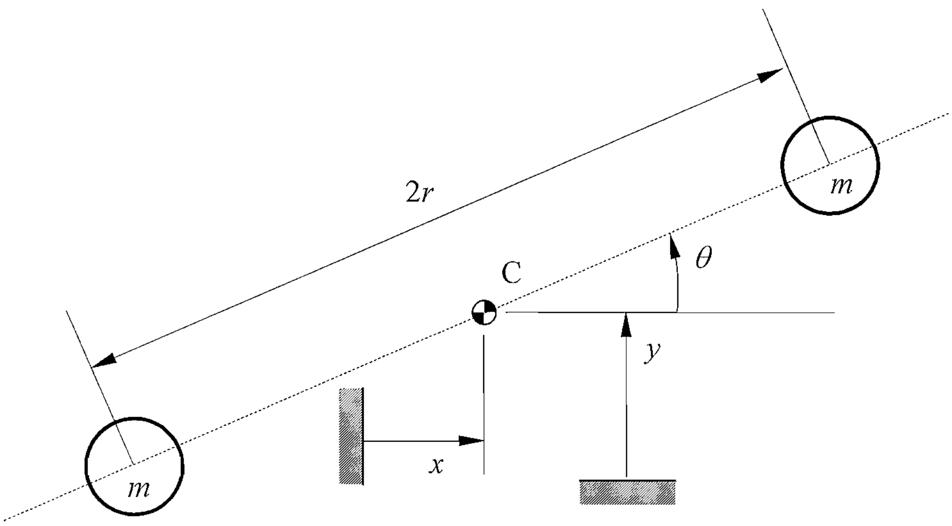

Fig. 4

5 A schematic of a pulley system is shown in Fig. 5. An elastic belt connects two pulleys A and B each of mass m and radius r. The centre of pulley A is fixed in space while the centre of pulley B is constrained to move vertically. The rotation of pulleys A and B are described by angles θ_1 and θ_2 respectively and the displacement of pulley B is x. Each pulley may be taken to have a polar moment of inertia mr^2. For small oscillations of the system each side of the belt can be represented as a spring of stiffness k as shown.

(a) Neglecting any gravitational forces, use Lagrange's equations to derive the mass and stiffness matrices of the system. [50%]

(b) It is evident that the system has a vibration mode with a natural frequency of zero, which involves only rotation of the pulleys. Write down the eigenvector for this mode and demonstrate that it satisfies the matrix equations of motion. [20%]

(c) Pulley A is now constrained completely so that $\theta_1 = 0$. Write down the matrix equations of motion that govern the two remaining degrees of freedom and determine the natural frequencies and mode shapes. [30%]

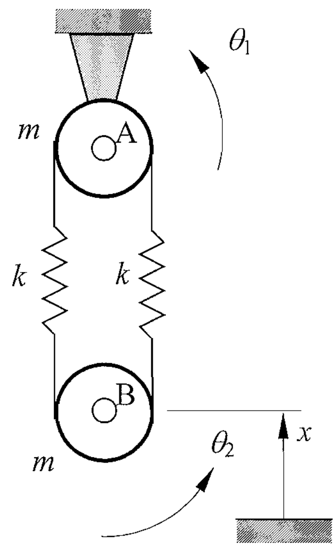

Fig. 5

END OF PAPER

Part IIA Data sheet
Module 3C5 Dynamics
Module 3C6 Vibration

Dynamics in three dimensions

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{dp}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity ω about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\omega \times \boldsymbol{r})\, dm \;=\; \boldsymbol{I}\,\omega$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$A = \int (y^2 + z^2)dm \qquad B = \int (z^2 + x^2)dm \qquad C = \int (x^2 + y^2)dm$$
$$D = \int yz\, dm \qquad E = \int zx\, dm \qquad F = \int xy\, dm$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \Omega \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^{\ 2} M\underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

2. Kinetic energy

$$T = \frac{1}{2}\underline{\dot{u}}^t M \underline{\dot{u}}$$

3. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\, \underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and a_n gives the "amount" of the nth mode.

5. Transfer function

For (generalised) force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\, u_n(x)$$

where $w(x,t)$ is the displacement and a_n gives the "amount" of the nth mode.

For force F at frequency ω applied at point x, and response w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.