ENGINEERING TRIPOS PART IIA

Friday 12 May 9.00 to 10.30

Module 3C6

VIBRATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)

STATIONERY REQUIREMENTS
Single-sided paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 A uniform solid circular shaft of radius a undergoes free torsional vibration. The material of the shaft has shear modulus G and density ρ.

(a) By considering a small element of the shaft, derive the differential equation that governs the vibration, in the form given in the Data Sheet. [25%]

(b) Such a shaft, of length L, is built-in to a rigid wall at one end, and is free at the other end. Write down boundary conditions for each end of the shaft and hence find the natural frequencies and mode shapes of the shaft. [25%]

(c) A rigid rotor with polar moment of inertia K is now attached to the free end of the shaft. Write down the boundary condition which now applies at this end of the shaft, and hence obtain an equation satisfied by the natural frequencies of the shaft/rotor system. [25%]

(d) Sketch a graphical solution of the equation in part (c). Comment on the physical interpretation of the pattern of natural frequencies revealed. [25%]

2 A beam of length L, mass per unit length m and bending rigidity EI is clamped at both ends, which lie at $x = 0$ and $x = L$. The beam undergoes small transverse vibration with displacement $w(x,t)$.

(a) Starting from the differential equation given in the Data Sheet, outline without detailed calculation, the sequence of steps involved in calculating the natural frequencies and mode shapes. Sketch the first three mode shapes. [30%]

(b) You may assume that the natural frequencies ω_n are the solutions for ω of the equation

$$\cos\alpha L \cosh\alpha L = 1$$

where

$$\alpha^4 = \frac{m\omega^2}{EI}.$$

Sketch a graphical solution to this equation, and deduce an approximate formula for ω_n.

[20%]

(c) Consider the pattern of transverse displacement

$$w = x^2(x - L)^2.$$

Sketch this function and comment on its suitability for an assumed mode shape. Use this assumed displacement in Rayleigh's principle to estimate the lowest natural frequency.

You may assume that $\int_0^L w^2 dx = \frac{L^9}{630}$ with the given function w.

Is the estimate consistent with your answer from (b)? [50%]

3 Two uniform disks '1' and '2', of radius R and mass m roll without slip on a horizontal table as shown in Fig. 1. They are connected together and to a rigid wall by two springs of stiffness k, through frictionless bearings at the centre of each disk. The displacements of the two disks from equilibrium are y_1 and y_2.

(a) Write expressions for the kinetic and potential energies of the system. Hence derive the mass and stiffness matrices. [25%]

(b) Calculate the natural frequencies and natural mode shapes of the system. [25%]

(c) Disk 1 is rolled (without slip) clockwise through 45° from its equilibrium position while disk 2 is held in its equilibrium position. The two disks are then released simultaneously from rest. Calculate the angle of rotation *from its equilibrium position* of disk 1 at time $\sqrt{m/k}$ after the release. [25%]

(d) The stiffness of the spring connecting the two disks is increased by 20%. Use Rayleigh's principle to revise your answer to part (c) for the new system. [25%]

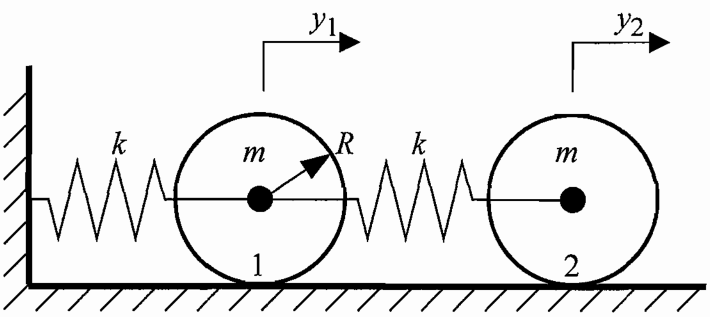

Fig. 1

4 Torsional vibration of the crankshaft of an engine is represented by the model shown in Fig. 2. Five disks, each with polar moment of inertia J are connected by four massless shaft sections, each with torsional stiffness k. The assembly is supported by frictionless bearings. The angular positions of the disks are defined by the coordinate vector $[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]^T$.

(a) Write down expressions for the kinetic and potential energies of the system. [10%]

(b) Without detailed calculation, sketch the mode shapes. Explain salient features. [25%]

(c) Estimate the lowest non-zero natural frequency using Rayleigh's principle. [25%]

(d) A sinusoidal torque is applied to disk 1. Sketch the magnitude of the response of disk 3 as a function of the frequency of input. Use a dB scale. [30%]

(e) As a model of a component of a real engine, discuss the features that are missing and their likely effects on the response of the system. [10%]

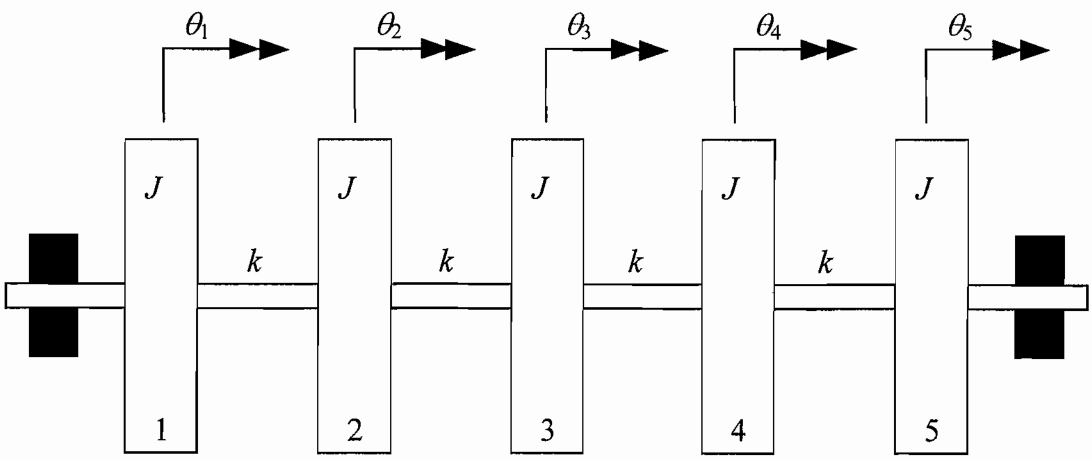

Fig. 2

END OF PAPER

Part IIA Data sheet

Module 3C5 Dynamics

Module 3C6 Vibration

Dynamics in three dimensions

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i :

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity ω about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and

$$A = \int (y^2 + z^2)\,dm \qquad B = \int (z^2 + x^2)\,dm \qquad C = \int (x^2 + y^2)\,dm$$
$$D = \int yz\; dm \qquad E = \int zx\; dm \qquad F = \int xy\; dm$$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B - C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C - A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A - B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $\boldsymbol{Q} = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^{\ 2} M\underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{u}}^t M \dot{\underline{u}}$$

3. Orthogonality and normalisation

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

4. General response

The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\, \underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and a_n gives the "amount" of the nth mode.

5. Transfer function

For (generalised) force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

Kinetic energy

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

Orthogonality and normalisation

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

General response

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\, u_n(x)$$

where $w(x,t)$ is the displacement and a_n gives the "amount" of the nth mode.

Transfer function

For force F at frequency ω applied at point x, and response w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{q}{F} \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

7. Impulse response

For a unit impulse applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$g(j,k,t) = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(j,k,t) \approx \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t$$

(with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin \omega_n t \; e^{-\omega_n \zeta_n t}$$

(with small damping).

8. Step response

For a unit step force applied at $t = 0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$h(j,k,t) = \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(j,k,t) \approx \sum_n u_j^{(n)} u_k^{(n)} \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t\right]$$

(with no damping), or

$$h(t) \approx \sum_n u_n(x)\, u_n(y) \left[1 - \cos \omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

(with small damping).

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi / 2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

ENGINEERING TRIPOS PART IIB

Module 3C6 Examination, 2006

Answers

1 (b) $x = 0,\ \theta = 0;\ x = L,\ d\theta/dx = 0.\quad \omega_n = \sqrt{\frac{G}{\rho}}\left(n+\frac{1}{2}\right)\frac{\pi}{L};\quad u_n(x) = \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L}$

(c) $K\frac{\partial^2\theta}{\partial t^2} = -GJ\frac{\partial\theta}{\partial x}$ at $x = L$; $\tan kL = \frac{J\rho}{Kk}$, with $k^2 = \frac{\rho\omega^2}{G}$

2 (b) $\omega_n^2 \approx \left(n+\frac{1}{2}\right)^4 \pi^4 \frac{EL}{mL^4}$; (c) $\omega^2 = 504\frac{EI}{mL^4}$

3 (a) $T = \frac{1}{2}\left(\frac{3}{2}m\dot{y}_1^2 + \frac{3}{2}m\dot{y}_2^2\right)$; $V = \frac{1}{2}ky_1^2 + \frac{1}{2}k\left(y_2 - y_1\right)^2$; $\frac{3m}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; $k\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}$

(b) $\omega^2 = \frac{k}{m}\left(1 \pm \frac{\sqrt{5}}{3}\right)$; $\begin{bmatrix}0.618 & 1\end{bmatrix}^T$; $\begin{bmatrix}-1.618 & 1\end{bmatrix}^T$

(c) 0.33 rads; (d) 0.28 rads

4 (a) $T = \frac{1}{2}J\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2 + \dot{\theta}_4^2 + \dot{\theta}_5^2\right)$

$$V = \frac{1}{2}k\left[\left(\theta_2 - \theta_1\right)^2 + \left(\theta_2 - \theta_1\right)^2 + \left(\theta_3 - \theta_2\right)^2 + \left(\theta_4 - \theta_3\right)^2 + \left(\theta_5 - \theta_4\right)^2\right]$$

(c) $\omega_1^2 \approx 0.4\frac{k}{J}$