ENGINEERING TRIPOS PART IIA

Thursday 11 May 2006 9.00 to 10.30

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Special datasheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Discuss the conditions under which a thin circular disk has circular symmetry. [15%]

(b) Starting from the general expressions of equilibrium provided on the datasheet, show that for a thin circular disk with circular symmetry, the equilibrium equation expressed in polar coordinates (r,θ) is given by

$$r\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r}=\sigma_{\theta\theta}-\sigma_{rr}$$

where σ_{rr} and $\sigma_{\theta\theta}$ are the radial and hoop stresses, respectively. [15%]

(c) Starting from the general compatibility equations provided on the datasheet, show that for the circular disk described in (b), the compatibility equations reduce to

$$r\frac{\mathrm{d}\varepsilon_{\theta\theta}}{\mathrm{d}r}=\varepsilon_{rr}-\varepsilon_{\theta\theta}$$

where ε_{rr} and $\varepsilon_{\theta\theta}$ are the radial and hoop strains, respectively. [25%]

(d) Consider a thin circular plate with a central hole of radius a. The plate is made of material with Young's modulus E, Poisson ratio ν and coefficient of thermal expansion α. Upon non-uniform heating, the distribution of temperature in the plate, initially at uniform temperature T_0, is $T(r)$. Express the stress versus strain relationships for the heated plate in polar coordinates. [15%]

(e) With u denoting the radial displacement, the equilibrium equation for the hollow circular plate described in (d) is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{d(ur)}{dr}\right]=(1+\nu)\alpha\frac{\mathrm{d}(T-T_0)}{\mathrm{d}r}.$$

Solve for the displacements in the heated plate. [30%]

2 (a) A thin plate with a small triangular protrusion on one side is subjected to uniform tension σ at the ends, as shown in Fig. 1(a). Under plane stress conditions, show that the triangular protrusion is stress free (you may assume that the stress state in the remainder of the plate is unaffected by the presence of the triangular protrusion). [20%]

(b) The tensile stress σ acting on the plate of Fig. 1(a) is replaced by an in-plane hydrostatic pressure p.

(i) Assuming plane stress, show that the stress field

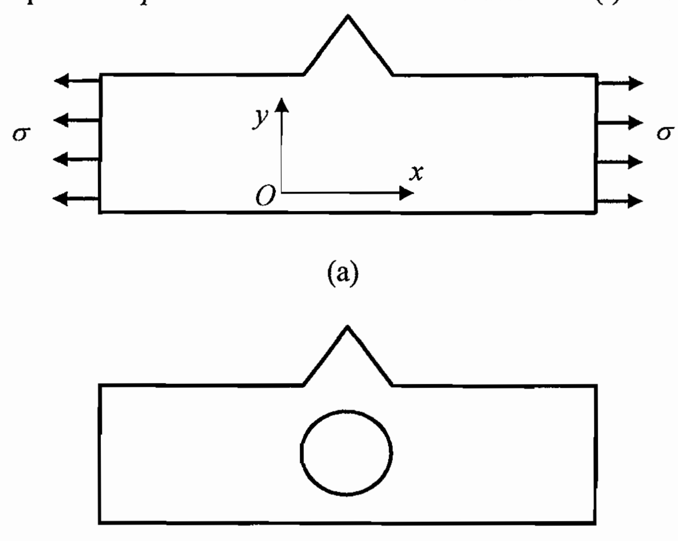
$$\sigma_{xx} = \sigma_{yy} = -p, \quad \sigma_{xy} = 0$$

satisfies both the equilibrium equations as well as the boundary conditions. [25%]

(ii) Find the strain field in the plate, and hence the displacement field. [30%]

(iii) Show that the strain field obtained in (ii) satisfies the compatibility condition. [10%]

(iv) A circular hole is now drilled in the plate of Fig. 1(a) as shown in Fig. 1(b). In addition to the in-plane pressure p acting on the external surface of the plate, the surface of the hole is also subjected to a uniform in-plane pressure p. Comment on how the solutions in (i) and (ii) will change. [15%]



(a)

(b)

Fig. 1

(TURN OVER

3 A light uniform cantilever of depth $2b$ and length L is shown in Fig. 2. This cantilever is loaded by a uniform normal force per unit length q on the top and bottom surfaces. A candidate Airy stress function for representing the stress field in this cantilever is

$$\phi = \frac{A}{6}x^2y^3 + \frac{B}{3}x^3y + \frac{C}{2}x^2y + \frac{D}{20}y^5$$

where A, B, C and D are constants.

(a) Determine the relationship between A and D for ϕ to be a valid Airy stress function. [15%]

(b) For a stress field that satisfies the boundary conditions, determine the constants A, B, C and D in terms of q and b. Hence also determine the stresses σ_{xx}, σ_{yy} and σ_{xy}. [40%]

(c) Show that the stress field obtained in (b) gives the correct shear force distribution along the length of the cantilever. [20%]

(d) Determine the bending moment due to the stress field obtained in (b) at $x = 0$. Hence, appropriately modify the stress field obtained in (b) in order to satisfy boundary conditions at the free end of the cantilever. [25%]

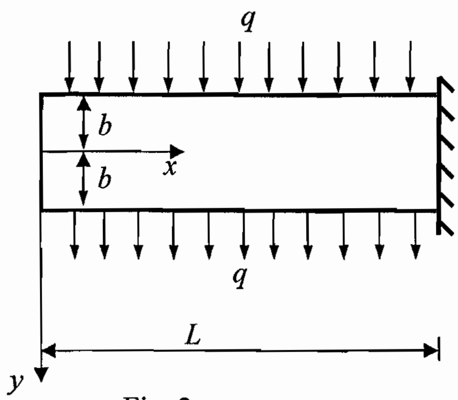


Fig. 2

4 (a) State briefly

(i) the upper bound theorem; and [15%]

(ii) the lower bound theorem [15%]

of plasticity theory.

(b) Consider a square plate of side $2b$ containing a central circular hole of radius a as sketched in Fig. 3. The plate is made from a rigid ideally-plastic Tresca material with a tensile yield strength Y. A uniform in-plane pressure p is applied to the surface of the hole. Assume plane strain conditions and an axi-symmetric displacement field with a radial displacement

$$u = \frac{A}{r}$$

where A is a constant and r the radial ordinate measured from the centre of the plate.

(i) Calculate the external work done by the applied pressure. [20%]

(ii) By considering a 45° segment of the plate, calculate the internal plastic dissipation in the plate, and hence determine an upper bound to the pressure p required to collapse the plate.

[Hint: $\int_0^{\pi/4} \ln(\cos\theta)\, d\theta = -(\pi/4)\ln 2 + 0.458$] [40%]

(iii) Briefly discuss the accuracy of the collapse pressure calculated above. [10%]

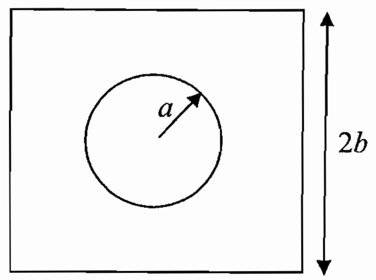


Fig. 3

END OF PAPER

Paper G4: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2\gamma_{xy}}{\partial x\partial y} = \dfrac{\partial^2\varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2\varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial\gamma_{r\theta}}{\partial\theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial\varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial\varepsilon_{rr}}{\partial r} + \dfrac{\partial^2\varepsilon_{rr}}{\partial\theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial\sigma_{r\theta}}{\partial\theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial\sigma_{yy}}{\partial y} + \dfrac{\partial\sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right\} \times \left\{\dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2\phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial\phi}{d\theta}\right\}$

3. Torsion of prismatic bars

Prandtl stress function: $\sigma_{zx}\ (= \tau_x) = \dfrac{dF}{dy}$, $\sigma_{zy}\ (= \tau_y) = -\dfrac{dF}{dx}$

Equilibrium: $T = 2\int_A F\,dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. Total potential energy of a body

$$\Pi = U - W$$

where $U = \frac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. Principal stresses and stress invariants

Values of the principal stresses, σ_P, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx} - \sigma_P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_P & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_P \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_P.

Expanding: $\sigma_P^3 - I_1\sigma_P^2 + I_2\sigma_P - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} .$$

6. Equivalent stress and strain

Equivalent stress $\bar{\sigma} = \sqrt{\frac{1}{2}}\,\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\frac{2}{3}}\,\{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. Yield criteria and flow rules

Tresca

Material yields when maximum value of $|\sigma_1 - \sigma_2|$, $|\sigma_2 - \sigma_3|$ or $|\sigma_3 - \sigma_1| = Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1 - \sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2 - \sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3 - \sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}} .$$

Answers to 3C7: Mechanics of Solids (2005-2006)

1. (e) $u = C_2 r + \frac{C_1}{r} + (1+\nu)\frac{\alpha}{r}\int_a^r r\Delta T dr$

3. (a) $D = -\frac{2A}{3}$

4. (b)(i) $W = 2\pi pA$

(b)(ii) $p_c = \frac{4Y}{\pi}\left(\frac{\pi}{4}\ln\frac{2b}{a} - 0.458\right)$