ENGINEERING TRIPOS PART IIA

Saturday 13 May 2006 9 to 10.30

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Special datasheets (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The two-dimensional pin-jointed structure shown in Fig. 1 consists of three bars, each with axial stiffness AE.

(a) Set up the stiffness matrix $\mathbf{K}$ which relates the nodal displacements of the structure $\mathbf{d} = [d_1 \quad d_2 \quad d_3 \quad d_4]^{\mathrm{T}}$ to the corresponding set of external loads $\mathbf{p} = [p_1 \quad p_2 \quad p_3 \quad p_4]^{\mathrm{T}}$. [40%]

(b) Show that $\mathbf{Kd} = \mathbf{p}$ has no solution for $\mathbf{p} = [1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$. Explain why this load cannot be equilibrated in the configuration shown. [30%]

(c) Compute the displacements $\mathbf{d}$ due to the loads $\mathbf{p} = [1 \quad 0 \quad -1 \quad 0]^{\mathrm{T}}$. Is $\mathbf{d}$ uniquely defined by the stiffness equations? [30%]

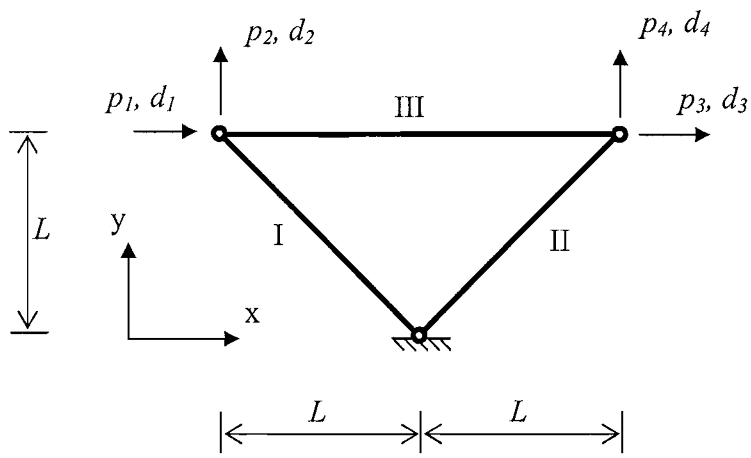

Fig. 1

2 (a) A finite element mesh of six-node triangular elements is used to model a thin, linear-elastic plate subject to in-plane loads and displacements. Explain what discontinuities should be expected in the displacement, stress and strain fields predicted by the finite element model. [20%]

(b) Figure 2 shows two elements from a mesh of the type described above. Sketch the shape functions corresponding to nodes A and B over both elements. [20%]

(c) Derive expressions for the displacement field over the two elements given that the nodal displacement components are: $d_{Ax} = 10^{-4}$ mm, $d_{Ay} = 0$ mm, $d_{Bx} = 10^{-4}$ mm, $d_{By} = -10^{-4}$ mm. All other nodal displacement components are zero. [20%]

(d) Compute the state of strain at points P_1 and P_2 which lie on either side of the boundary between elements 1 and 2. Both points have approximately the same coordinates (0 mm, 1.5 mm) but are within different elements. [20%]

(e) The plate has Young's Modulus $E = 200$ kN mm^{-2} and Poisson's ratio $\nu = 0.3$. Calculate the stress σ_{xx} acting on either side of a small section of plate which contains both points P_1 and P_2 and lies across the boundary of elements 1 and 2. Is this section in equilibrium? [20%]

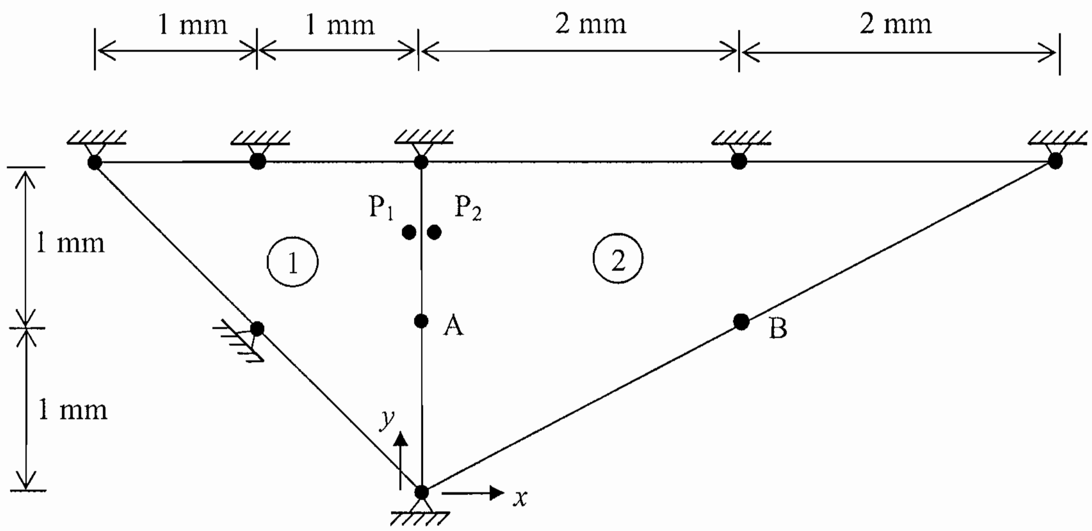

Fig. 2

(TURN OVER

3 (a) Groundwater flow in a rock formation is considered. The water flow rate per unit area in the i-direction, q_i, is expressed as follows:

$$q_i = -k_i \frac{\partial h}{\partial i}$$

where h is the hydraulic head and k_i is the rock permeability in the i-direction. Show that the governing equation of three dimensional groundwater flow in a rigid rock formation can be expressed as follows.

$$k_x \frac{\partial^2 h}{\partial x^2} + k_y \frac{\partial^2 h}{\partial y^2} + k_z \frac{\partial^2 h}{\partial z^2} + Q = 0$$

where Q is the source of water per unit volume. [20%]

(b) For the case of one-dimensional flow in the x-direction, show that the weak form of the governing equation is:

$$\int_b^a \frac{dv}{dx} k_x \frac{dh}{dx} dx = (v)_{x=b} q_b - (v)_{x=a} q_a + \int_b^a vQdx$$

where v is a weight function. The flow rate at the end boundaries is $q = q_a$ at $x = a$ and $q = q_b$ at $x = b$. [15%]

(c) The hydraulic head h and the weight function v are approximated using the following shape functions:

$$h = \mathbf{Na}, \quad \frac{dh}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{a} = \mathbf{Ba}$$

$$v = \mathbf{Nc}, \quad \frac{dv}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{c} = \mathbf{Bc}$$

(cont.

where **N** and **B** are the shape function matrices, **a** is the nodal hydraulic head values in vector form and **c** is the arbitrary nodal values in vector form. Show the finite element approximation of the weak form given in (b). [15%]

(d) Groundwater is pumped from a rock aquifer using a well. The layered rock formation at the site is shown in Fig. 3. The well extends down to the middle of rock layer 2. The well in rock layer 1 is cased; that is, water cannot flow into the well in this layer. Measurements on rock samples taken from the ground show that the vertical permeability is different from the horizontal permeability. Sketch a finite element mesh that can be used to compute the steady state hydraulic head distribution inside the rock formations. In order to reduce computational time, consider symmetry as much as possible. [25%]

(e) For the finite element calculation, the pressure inside the well is prescribed. Define the boundary conditions of the finite element mesh sketched in (d). Explain how to evaluate the pumping rate from the computed results. [25%]

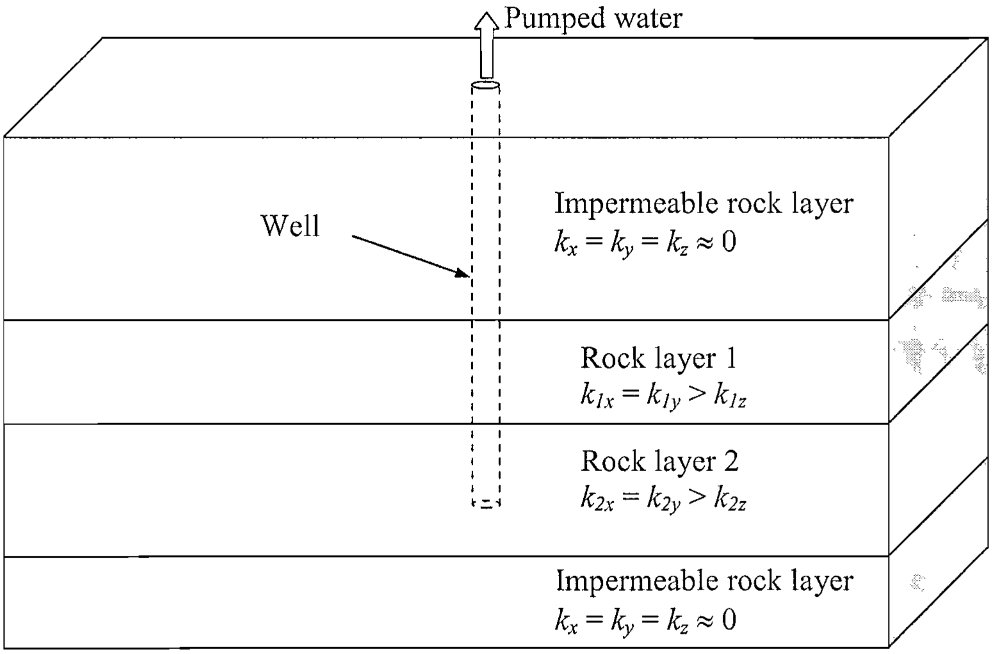

Fig. 3

4 (a) Figure 4 shows a simple finite element mesh of a thin plate of thickness t consisting of only two six-node triangular elements. A uniform pressure p is applied to one side of the plate as shown. Find the equivalent nodal loads at the nodes 1, 2 and 3. [50%]

(b) Figure 5 shows a portion of a simple finite element mesh for the analysis of the stress distribution in a thin plate of thickness t containing a circular hole of radius R. The mesh is a one quarter model of the plate and is made up of six-node triangular elements. The edge of the circular hole is represented by the straight, and equal length, lines ABC, CDE and EFG. If the hole is loaded by an internal pressure p, find the equivalent nodal loads at the nodes A to G. [50%]

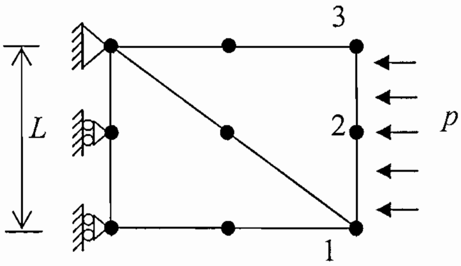

Fig. 4

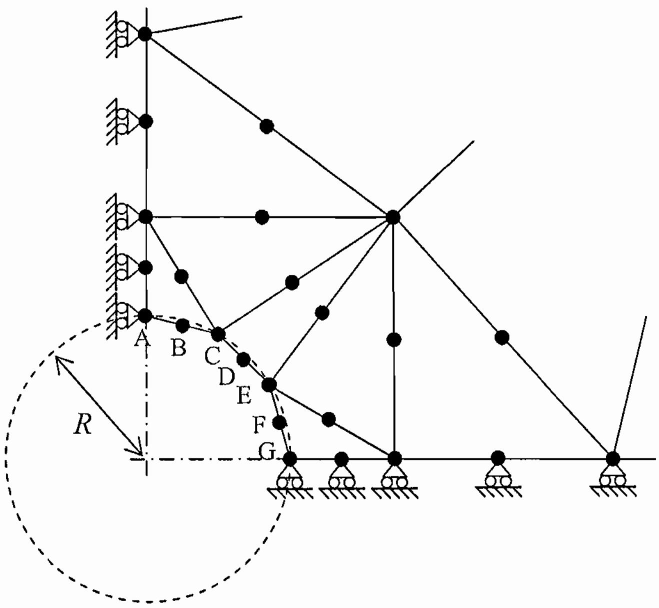

Fig. 5

END OF PAPER

Part IIA: Module 3D7 2003-4

Finite Element Methods

Formulae

Force Method

Stress resultants:	solve $\mathbf{Hr} = \mathbf{p}$ and find $\mathbf{r} = \mathbf{r}_0 + \mathbf{Sx}$;
	then, solve $\mathbf{S}^T\mathbf{FSx} = -\mathbf{S}^T(\mathbf{Fr}_0 + \mathbf{e}_0)$ for $\mathbf{x}$.
Displacements:	solve $\mathbf{H}^T\mathbf{d} = \mathbf{e}$, where $\mathbf{e} = \mathbf{Fr} + \mathbf{e}_0$.

Displacement Method

Displacements:	solve $\mathbf{Kd} = \mathbf{p}$.
Stress resultants:	for element i, solve $\mathbf{F}_i\mathbf{r}_i = \mathbf{e}_i$, where $\mathbf{e}_i = (\mathbf{H}_i')^T\mathbf{d}_i'$.

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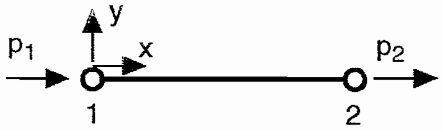
February 2004

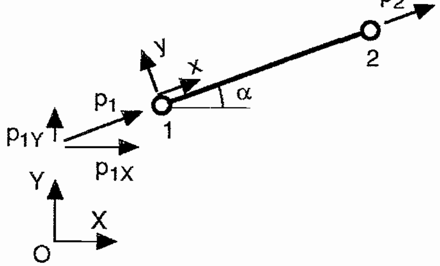
PIN-JOINTED BAR in LOCAL COORDINATES



Static variables	Kinematic variables	Equilibrium	Elasticity	Stiffness $\mathbf{K}_i = \mathbf{H}_i \mathbf{F}_i^{-1} \mathbf{H}_i^T$
$\mathbf{r}_i = [t]$ t = axial force $\mathbf{p}_i = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$	$\mathbf{e}_i = [e]$ e = extension $\mathbf{d}_i = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$	$\mathbf{H}_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\mathbf{F}_i = [a]$	$\mathbf{K}_i = \begin{bmatrix} 1/a & -1/a \\ -1/a & 1/a \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness	$\mathbf{H}_i \mathbf{r}_i = \mathbf{p}_i$ $\mathbf{H}_i^T \mathbf{d}_i = \mathbf{e}_i$ $\mathbf{F}_i \mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}_i \mathbf{d}_i = \mathbf{p}_i$			

$a = L/AE$, AE = axial stiffness

PIN-JOINTED BAR in GLOBAL COORDINATES



Static variables	Kinematic variables	Coordinate transformation	Equilibrium	Stiffness
$\mathbf{r}_i$ $\mathbf{p}_i' = \begin{bmatrix} p_{1X} \\ p_{1Y} \\ p_{2X} \\ p_{2Y} \end{bmatrix}$	$\mathbf{e}_i$ $\mathbf{d}_i' = \begin{bmatrix} d_{1X} \\ d_{1Y} \\ d_{2X} \\ d_{2Y} \end{bmatrix}$	$\mathbf{T}_i = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$ $\mathbf{R} = \begin{bmatrix} u \\ v \end{bmatrix}$	$\mathbf{H}_i' = \begin{bmatrix} -u \\ -v \\ u \\ v \end{bmatrix}$	$\mathbf{K}_i' = \frac{1}{a} \begin{bmatrix} u^2 & uv & -u^2 & -uv \\ & v^2 & -uv & -v^2 \\ & & u^2 & uv \\ \text{symm.} & & & v^2 \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness Transformations	$\mathbf{H}_i' \mathbf{r}_i = \mathbf{p}_i'$ $\mathbf{H}_i'^T \mathbf{d}_i' = \mathbf{e}_i$ $\mathbf{F}_i \mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}_i' \mathbf{d}_i' = \mathbf{p}_i'$ $\mathbf{T}_i \mathbf{p}_i = \mathbf{p}_i'$ $\mathbf{T}_i \mathbf{d}_i = \mathbf{d}_i'$ $\mathbf{T}_i \mathbf{H}_i = \mathbf{H}_i'$ $\mathbf{T}_i \mathbf{K}_i \mathbf{T}_i^T = \mathbf{K}_i'$			

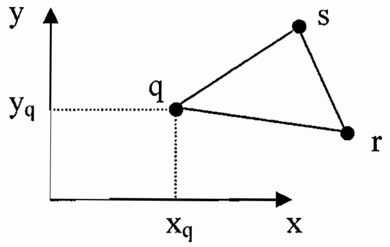
$a = L/AE$, AE = axial stiffness, $u = \cos\alpha$, $v = \sin\alpha$

Basic relationships, for element j :

displacements	$\mathbf{u} = \mathbf{N}^j \mathbf{d}^j$
strains	$\boldsymbol{\varepsilon} = \mathbf{B}^j \mathbf{d}^j$
stresses	$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\,\mathbf{B}^j \mathbf{d}^j$
stiffness matrix	$\mathbf{K}^j = \int (\mathbf{B}^j)^T \mathbf{D}\,\mathbf{B}^j\, dV$
stiffness equations	$\mathbf{K}^j \mathbf{d}^j = \mathbf{p}^j$

Material stiffness (for plane stress) $\mathbf{D} = \dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$

Shape functions of some simple plane stress elements

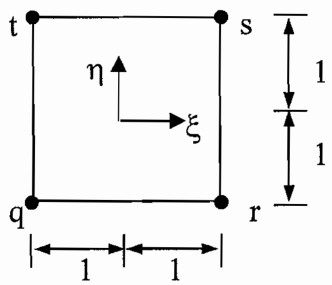


$n_q = [(x_r y_s - x_s y_r) + (y_r - y_s)x + (x_s - x_r)y]/2A$

$n_r = [(x_s y_q - x_q y_s) + (y_s - y_q)x + (x_q - x_s)y]/2A$

$n_s = [(x_q y_r - x_r y_q) + (y_q - y_r)x + (x_r - x_q)y]/2A$

A = area of triangle

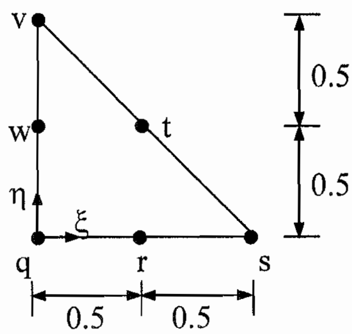


$n_q = (1 - \xi)(1 - \eta)/4$

$n_r = (1 + \xi)(1 - \eta)/4$

$n_s = (1 + \xi)(1 + \eta)/4$

$n_t = (1 - \xi)(1 + \eta)/4$



$n_q = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$

$n_r = 4\xi\,(1 - \xi - \eta)$

$n_s = \xi\,(2\xi - 1)$

$n_t = 4\,\xi\,\eta$

$n_v = \eta\,(2\eta - 1)$

$n_w = 4\eta\,(1 - \xi - \eta)$

3D7 Answers

1. (a) $$\frac{AE}{2\sqrt{2}L}\begin{bmatrix} 1+\sqrt{2} & -1 & -\sqrt{2} & 0 \\ -1 & 1 & 0 & 0 \\ -\sqrt{2} & 0 & 1+\sqrt{2} & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) –

(c) $$\frac{L}{EA}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}x \quad \text{or} \quad \frac{L}{EA}\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}x$$

2. (a) –

(b) –

(c) Element 1: $u_1 = (2-x)(x+y)\times 10^{-4}$, $v_1 = 0$,
Element 2: $u_2 = (2-y)y\times 10^{-4}$, $v_2 = (1/2)x(y-2)\times 10^{-4}$

(d) Element 1: $\varepsilon_x = 0.5\times 10^{-4}$, $\varepsilon_y = 0$, $\gamma_{xy} = -1\times 10^{-4}$
Element 2: $\varepsilon_x = 0$, $\varepsilon_y = 0$, $\gamma_{xy} = -1.25\times 10^{-4}$

(e) Element 1: $\sigma_x = 10.99$ N/mm^2
Element 2: $\sigma_x = 0$

3. (a)&(b) –

(c) $$\left(\int_b^a \mathbf{B}^T k_x \mathbf{B} dx\right)\mathbf{a} = \left(\mathbf{N}^T\right)_{x=b} q_b - \left(\mathbf{N}^T\right)_{x=a} q_a + \int_b^a \mathbf{N}^T Q dx$$

(d) & (f) –

4. (a) $p_{x1} = -(1/6)tpL$, $p_{x2} = -(4/6)tpL$, $p_{x3} = -(1/6)tpL$, $p_{y1} = p_{y2} = p_{y3} = 0$

(b) Node A, 0.0833tpR, 90° from x axis
Node B, 0.345tpR, 75° from x axis
Node C, 0.167tpR, 60° from x axis
Node D, 0.345tpR, 45° from x axis
Node E, 0.167tpR, 30° from x axis
Node F, 0.345tpR, 15° from x axis
Node G, 0.0833tpR, 0° from x axis