ENGINEERING TRIPOS PART IIA

Tuesday 2 May 2006 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the feedback system

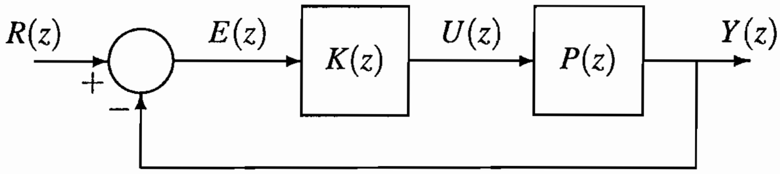

Fig. 1

where

$$P(z) = \frac{-1}{z^2 + z + 2}$$

and $K(z) = k$, which is a constant.

(a) Draw the pole/zero diagram for the open–loop system $P(z)$. Is the system stable? [20%]

(b) Find the closed–loop transfer function from $R(z)$ to $Y(z)$. [10%]

(c) For which values of k is the closed-loop system stable? [30%]

(d) Consider the closed–loop system and let the input r_n be an unit step. Find, as a function of k, the steady–state value of y_n (i.e., the $\lim_{n\to\infty} y_n$) when this is finite, stating for which values of k the answer is valid. [20%]

(e) Let $k = 1.5$. Fig. 2 shows three Nyquist plots (A, B and C), but only one corresponds to $kP(z)$. Choose the correct one, justifying your answer with respect to the Nyquist stability criterion. [20%]

(cont.

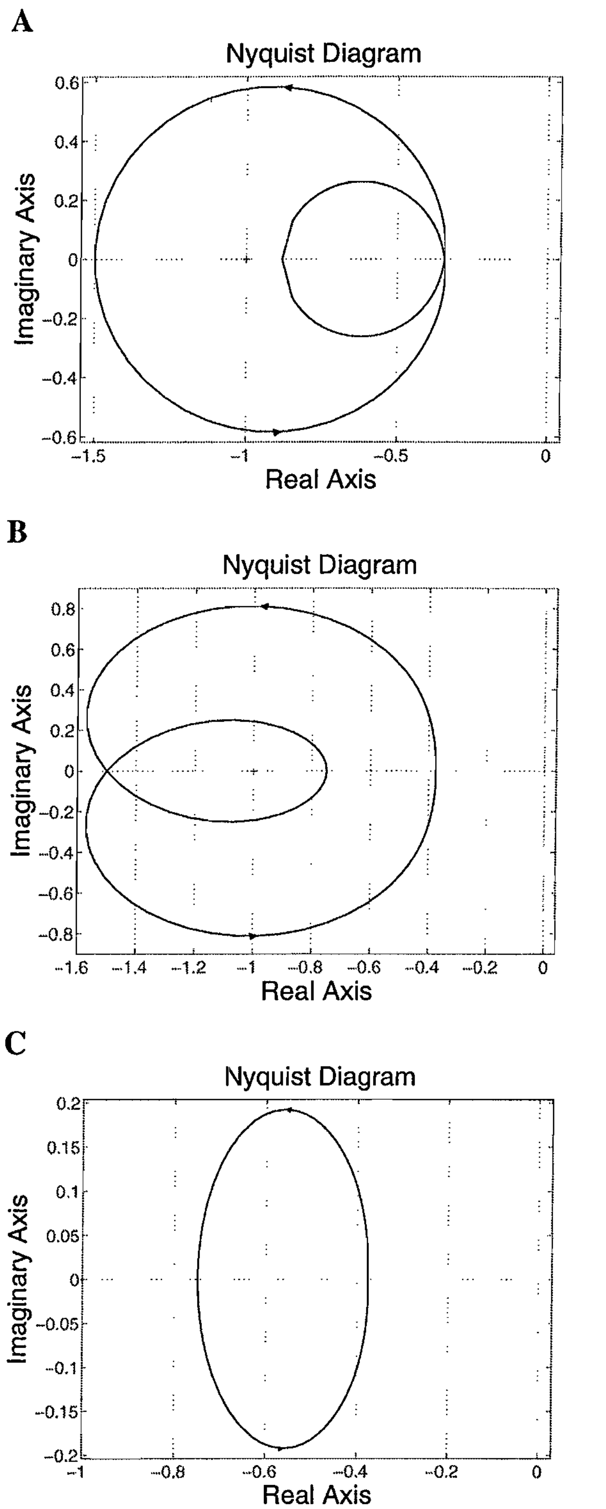

Fig. 2

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2 (a) The step response of a system is given by

$$y_n = \begin{cases} 0 & n \leq -1 \\ 1 & n = 0 \\ 0 & n = 1 \\ 1 & n = 2 \\ -1 & n = 3 \\ 0 & n \geq 4 \end{cases}$$

(i) Find the pulse response of the system. [20%]

(ii) Is the system causal? Is the system stable? Explain. [10%]

(iii) Find the input that results in the following response

$$x_n = \begin{cases} 0 & n \leq -1 \\ 1 & n = 0 \\ -2 & n = 1 \\ 1 & n = 2 \\ -3 & n = 3 \\ 2 & n = 4 \\ 0 & n \geq 5 \end{cases}$$

writing it in the form $\alpha_0 u_n + \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \cdots$, where u_n is the unit step. [20%]

(b) One random variable Y is a function of another random variable X such that $Y = g(X)$ and g is a strictly monotonically increasing function.

(i) Explain why the cumulative distribution functions of X and Y are related by

$$F_Y(y) = F_X(x) \qquad \text{if} \quad y = g(x) \; .$$

[10%]

(ii) The probability density function (pdf) of X is uniform over the range -1 to $+1$ and is zero elsewhere. It is desired that the pdf of Y should be Laplacian, given by

$$f_Y(y) = \frac{1}{2} \exp(-|y|)$$

Calculate the monotonically increasing function $g(X)$, that is required to generate Y from X. [40%]

3 (a) State the relationship between the auto-correlation function (ACF) of an ergodic random signal and its power spectral density (PSD). [15%]

(b) An ergodic random signal X has an ACF given by

$$r_{XX}(\tau) = P \operatorname{sinc}^2\left(\frac{\tau}{T_0}\right)$$

where P and T_0 are constants. Show that the PSD of this signal, $S_X(\omega)$, is of the form

$$S_X(\omega) = \begin{cases} \frac{2\pi P}{\omega_c}\left(1 - \frac{|\omega|}{\omega_c}\right) & \text{if } |\omega| < \omega_c \\ 0 & \text{if } |\omega| \geq \omega_c \end{cases}$$

and find ω_c as a function of T_0. (The E&I data book may be useful). [30%]

(c) The signal X is applied to a linear system and the PSD of the output signal Y is measured to be

$$S_Y(\omega) = \begin{cases} B(1 - |\omega| T_0) & \text{if } |\omega| < 1/T_0 \\ 0 & \text{if } |\omega| \geq 1/T_0 \end{cases}$$

Calculate $|H(\omega)|$, the magnitude of the frequency response of the system, for the angular frequency range $-2/T_0 < \omega < 2/T_0$. Why is it not feasible to calculate $|H(\omega)|$ outside of this range? [30%]

(d) If $H(\omega)$ is assumed to be zero outside of the range $-2/T_0 < \omega < 2/T_0$, explain why it would still be difficult to calculate the impulse response $h(t)$ of this system from the above measurements, and briefly describe a way to overcome this difficulty while still using X as the system input signal. [25%]

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4 Two coins are available. Coin A is biased with probability of Head equal 0.9. Coin B is fair. The two coins are tossed repeatedly in time according to the following rule: if the outcome at time t is Head then Coin A is tossed at time $t+1$, if the outcome at time t is Tail then Coin B is tossed at time $t+1$. A source X produces a sequence of symbols, taken from $\{H = \text{Head}, T = \text{Tail}\}$, according to the outcome of the toss at each time.

(a) Construct the table of conditional probabilities of X_{t+1} given X_t and find the equilibrium distribution of the random variable X_t. [20%]

(b) Calculate the joint entropy $H(X_{t+1}, X_t)$ and $H(X_{t+2}, X_t)$. Comment on the fact that $H(X_{t+2}, X_t) > H(X_{t+1}, X_t)$. [25%]

(c) Construct a Huffman code which encodes two symbols at a time. Calculate the average number of bits used to transmit one original symbol. [25%]

Someone claims to have constructed an optimal code which encodes more than 50 original symbols at a time and where 0.6 bits are used on average to encode one original symbol.

(d) Show that this value of average number of bits per original symbol does not violate the entropy bound. [15%]

(e) Can the claim of the optimality of such a code be correct? [15%]

END OF PAPER