ENGINEERING TRIPOS PART IIA

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Wednesday 3 May 2006 9 to 10.30

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Module 3F2

SYSTEMS AND CONTROL

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper
Graph paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 Consider a linear system in standard state-space form

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du \tag{1}$$

(a) Find an expression for the transfer function matrix relating the input vector $u$ to the output vector $y$. [20%]

(b) Explain briefly why every pole of the transfer function matrix is an eigenvalue of $A$. [20%]

(c) Figure 1 shows the headbox of a paper-making machine. It contains liquid pulp, which sprays onto a moving belt through the orifice at the bottom of the headbox. $H$ denotes the level of pulp in the headbox, and $P$ denotes the pressure of the air above the pulp (relative to ambient pressure). $M_i$ is the mass flow rate of pulp into the headbox, and $M_o$ is its mass flow rate onto the moving belt. $M_i$ and $P$ can be manipulated, to control $H$ and $M_o$. The level of pulp evolves according to the equation

$$\dot{H} = M_i - \sqrt{P + \frac{1}{5}H}$$

and the mass flow rate onto the belt is given by

$$M_o = \sqrt{P + \frac{1}{5}H}$$

The headbox is supposed to operate at a steady-state condition with $H = 1$ and $M_o = 1$. If $M^*$ and $P^*$ denote the corresponding steady-state values of $M_i$ and $P$, respectively, verify that $M^* = 1$ and $P^* = \frac{4}{5}$. [10%]

(d) If $H = 1 + x$, $M_o = 1 + y_1$, $M_i = M^* + u_1$, $P = P^* + u_2$, find a linearised model, in the form of equation (1), for small perturbations around the steady-state condition, with inputs $u_1$ and $u_2$, and outputs $y_1$ and $x$. [30%]

(e) Verify that the transfer function matrix corresponding to the linearised model found in part (d) is

$$G(s) = \frac{\begin{bmatrix} \frac{1}{10} & \frac{s}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}}{s + \frac{1}{10}}$$

[20%]

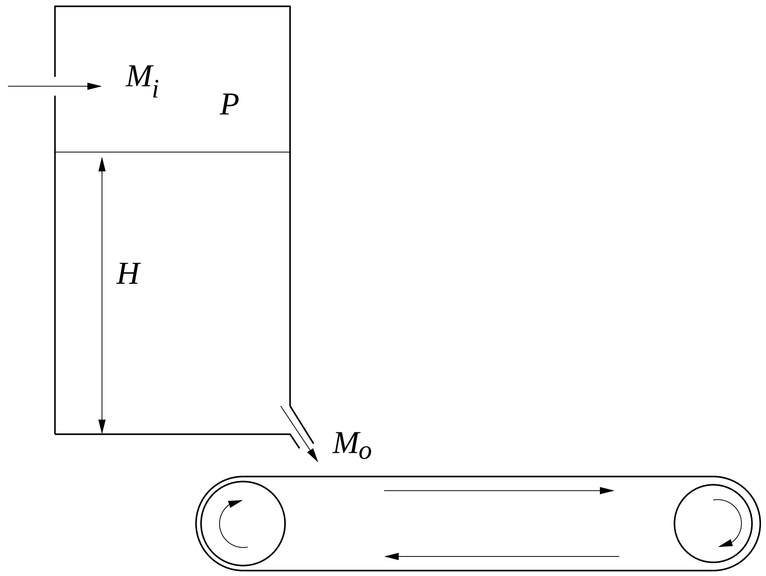


Fig. 1

2 In a machine-tool the transfer function from the amplifier input voltage to the speed of the tool head is

$$G(s) = \frac{1}{(s+2)(s^2+12s+40)} \qquad (2)$$

The speed is to be controlled by a feedback system.

(a) The root-locus diagram for this transfer function is shown in Fig.2. Estimate the *proportional gain* required to obtain two closed-loop poles located at the breakaway point at $-4$. [15%]

Where is the third closed-loop pole located with this proportional gain? [15%]

(b) Show that the *gain margin*, when the value of gain found in part (a) is used, is approximately 50. [30%]

(c) It is decided to use a *proportional and integral* (PI) controller with transfer function

$$K(s) = k\left(1 + \frac{3}{s}\right)$$

Sketch the root-locus diagram for the return-ratio $G(s)K(s)$ (assuming $k > 0$). Show any root-locus asymptotes on your sketch. (Accurate location of breakaway points is *not* required.) [30%]

(d) What would be the benefit of using the PI controller in this application, compared with a proportional gain only? [10%]

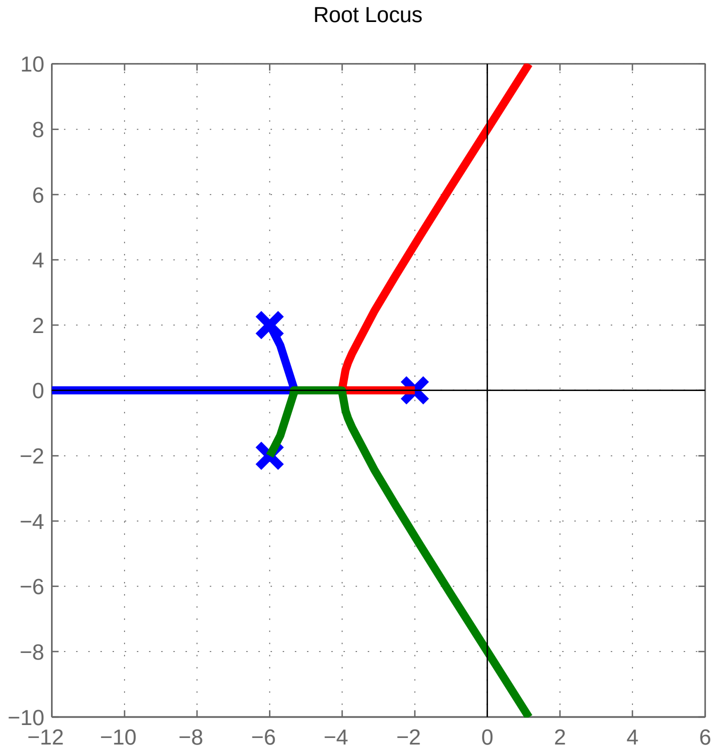


Fig. 2

3 (a) State a standard test for *controllability* of a linear system

$$\dot{x} = Ax + Bu \qquad (3)$$

[10%]

(b) Explain briefly why a linear system is not controllable if this test is not satisfied. [20%]

(c) In the 'ball and beam' apparatus the angle of the beam is denoted by $\theta$ and the position of the ball along the beam is denoted by $z$. For small angles it can be assumed that

$$\ddot{\theta} = \alpha u \quad \text{and} \quad \ddot{z} = \gamma\theta$$

where $u$ is the torque applied to the beam, and $\alpha$ and $\gamma$ are positive constants. Express the equations of motion of the system in the form of equation (3). [10%]

(d) Show that the ball and beam apparatus described in part (c) is controllable. [20%]

(e) It is suggested that the ball and beam apparatus described in part (c) can be controlled by feedback from the ball position and velocity, and the beam velocity only, omitting feedback from the beam position, so that the torque on the beam is of the form

$$u = -k_1\dot{\theta} - k_2\dot{z} - k_3 z$$

where $k_1$, $k_2$ and $k_3$ are constant gains. Show that it is *not* possible to obtain asymptotic stability with this scheme. [40%]

4 (a) Explain, with the aid of a suitable block-diagram, the purpose and operation of a *state observer*. [20%]

(b) A cart-mounted inverted pendulum makes an angle $\theta$ with the vertical, and the cart moves in a straight line with velocity $v$. The cart is driven by a force $f$. For small angles the linearised equations of motion are

$$\begin{aligned}\ddot{\theta} &= \theta + v + f \\ \dot{v} &= \theta - v - f\end{aligned}$$

Define a suitable state vector, keeping the state dimension as small as possible, and write these equations in standard state-space form. [20%]

(c) Show that the state in part (b) is observable if *either* $\theta$ *or* $v$ is measured (in addition to the force $f$). [20%]

(d) With reference to part (b), suppose that only $\theta$ and $f$ are measured. Design a state observer, locating all the observer poles at $-1$. [20%]

(e) If measurements of both $\theta$ and $v$ are available, discuss why it would be desirable to use both measurements for the estimation of the state, despite the result of part (c). [20%]

**END OF PAPER**

# 3F2 Systems and Control: 2006 Numerical answers

1. (a) —
   (b) —
   (c)

$$A = -\frac{1}{10}, \qquad B = [1, -\tfrac{1}{2}]$$
$$C = \begin{bmatrix} \frac{1}{10} \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

   (d) —

2. (a) Gain: 16. Third pole: −6.
   (b) Exact value is 51.
   (c) —
   (d) —

3. (a) —
   (b) —
   (c) If $x = [\theta, \dot{\theta}, z, \dot{z}]^T$ then

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \gamma & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \alpha \\ 0 \\ 0 \end{bmatrix}$$

   (d) —
   (e) —

4. (a) —
   (b) If $x = [\dot{\theta}, \theta, v]^T$ then

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

   (c) —
   (d)

$$L = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

   (e) —