ENGINEERING TRIPOS PART IIA

Saturday 13 May 2006 9 to 10.30

Module 3F3

SIGNAL AND PATTERN PROCESSING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The Discrete Fourier Transform (DFT) for a data sequence $\{x_n\}$ of length N is defined as

$$X_p = \sum_{n=0}^{N-1} x_n e^{-j\frac{2\pi}{N}np}, \; p = 0, 1, ..., N-1$$

(a) Describe the radix-2 Fast Fourier Transform (FFT) algorithm for implementation of the above DFT when N is a power of 2. Your description should include: an expression for the original length N DFT in terms of two length $N/2$ DFTs; the number of 'stages' in the FFT algorithm; bit reversal operations; in-place computation; and the butterfly structure.

What is the computational load (in complex multiplies and additions) for the FFT? [40%]

(b) Let $\{x_n\}$, $n = 0, 1, ..., N-1$, be a sequence of N data points with DFT $\{X_p\}$, where N is once again a power of 2. The sequence is known to satisfy

$$x_{n+N/2} = -x_n$$

for $n = 0, 1, \ldots, \frac{N}{2} - 1$.

Show that the DFT $\{X_p\}$ is zero for all *even* values of the frequency index p, i.e. show that $X_{2m} = 0$, for $m = 0, 1, \ldots, N/2-1$. [30%]

(c) Consider the modified sequence of $N/2$ data points

$$y_n = x_n \exp(-2jn\pi/N), \; n = 0, 1, ..., N/2-1$$

where $\{x_n\}$ is a sequence with properties as in part (b).

Write down an expression for the DFT $\{Y_p\}$ of the length $N/2$ sequence $\{y_n\}$. Starting from this expression, show that the non-zero values of the DFT $\{X_p\}$ in part (b) may be calculated as

$$X_{2m+1} = 2Y_m, \; m = 0, 1, ...N/2-1$$

[30%]

2 (a) Explain how the bilinear transform

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

can be used for the design of a digital filter from an analogue prototype filter. Your explanation should include the constraints on the type of filter that can usefully be designed, any distortions that are introduced, and procedures for transforming between one type of analogue prototype filter and another (low-pass to high-pass, for example). [40%]

(b) Show that the bilinear transform in (a) maps the unit circle in the z-plane onto the entire imaginary axis of the s-plane, according to a frequency warping function

$$\omega = \tan(\Omega/2)$$

where ω rads^{-1} is the frequency in the analogue domain and Ω rad/sample is the normalised digital frequency. [20%]

(c) In a high frequency audio perception experiment it is required to extract frequency components in a signal which lie *above* 20kHz, in a digital system having sampling rate 96 kHz. A low-pass Butterworth prototype filter is available having cut-off frequency 1 rads^{-1} and transfer function

$$H(s) = \frac{1}{(s^2 + \sqrt{2}s + 1)}.$$

Use this analogue prototype and the bilinear transform to obtain the transfer function of a possible digital filter for the experiment. Without additional calculations, comment on whether you would expect this filter to perform adequately in extracting high frequency components from the audio. [40%]

(TURN OVER

3 (a) For a discrete-time random process, define the terms autocorrelation function, wide sense stationarity, power spectrum and mean ergodicity. [20%]

(b) When a wide sense stationary random process is input to a linear, stable, time-invariant system, derive an expression for the autocorrelation function of the output random process, in terms of the input autocorrelation function and the impulse response of the linear system. Hence show that the output power spectrum can be expressed as

$$S_Y(e^{j\theta}) = |H(e^{j\theta})|^2 S_X(e^{j\theta})$$

where $S_X(e^{j\theta})$ is the input power spectrum and $H(e^{j\theta})$ is the frequency response of the linear system. [40%]

(c) A zero-mean white noise process $\{w_n\}$ is filtered through the system

$$y_n = 0.9 y_{n-1} - 0.81 y_{n-2} + w_n$$

Determine the poles and zeros of the system and its frequency response. Hence write down the power spectrum of y_n and sketch it, assuming that $E[w_n^2] = 1$. [40%]

4 Consider a study measuring the correlation between alcohol consumption and examination marks for Cambridge Undergraduates. The variable x measures units of alcohol consumed daily, while y measures marks out of 100. The following data are collected from 4 subjects in the study, $D = \{(0,80),(1,60),(3,60),(4,40)\}$, where each data point is an (x,y) pair.

Assume a linear regression model

$$y_i = ax_i + b + \varepsilon_i$$

where a and b are the slope and intercept of the regression line, respectively, and ε is zero mean Gaussian noise with variance σ^2. That is, $p(y_i|x_i,a,b,\sigma)$ is a Gaussian with mean $ax_i + b$ and variance σ^2, where $i \in \{1,\ldots,4\}$ indexes the subjects in the study.

Assume all the data points were generated independently from this model.

(a) Sketch the position of the points on a diagram and indicate the approximate position of the linear regression line. [20%]

(b) Write down the log likelihood as a function of the model parameters a, b, and σ. [20%]

(c) Assume $\sigma = 10$. Solve for the maximum likelihood settings of the parameters a and b. Your answer should show in detail the derivation steps involved in obtaining this solution, i.e. do not simply use your calculator. [40%]

(d) Given the values of a and b found in part (c), do you think that $\sigma = 10$ is a reasonable estimate, an overestimate, or an underestimate of σ? Justify your answer. [20%]

END OF PAPER