Part IIA 2007 – 3C3 – Solutions

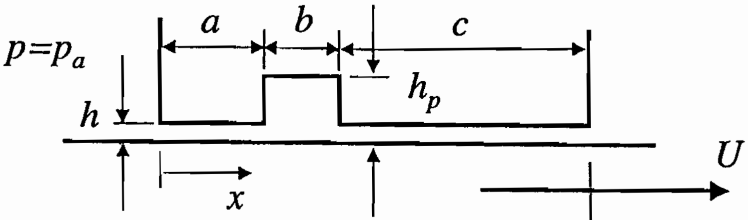
1 (a)

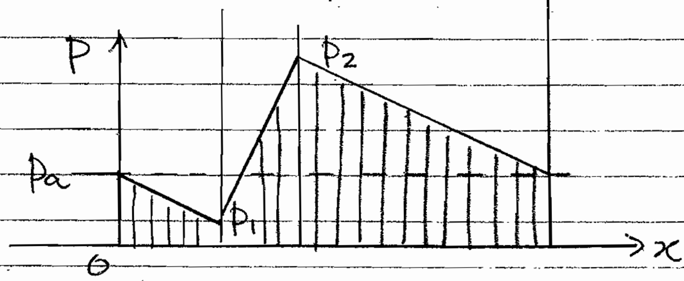
$$q = \frac{uh}{2} - \frac{h^3}{12\eta}\frac{dp}{dx}$$

Full derivation not expected although provided by some candidates. Enough to say that first Couette term associated with shear flow and the second Poisseuille term with pressure gradient. The velocity profiles being respectively linear and parabolic.

(b) (i) If h, U and η are same in each section, then since continuity of an incompressible fluid requires q to be constant it follows from the given equation that $\frac{dp}{dx}$ must have same value

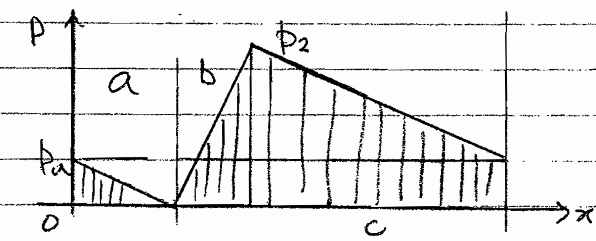
(ii)





possible for shaded area > $p_a(a+b+c)$ so some load carrying capacity.

(iii) If $p_1 \Rightarrow 0$ then profile becomes



Continuity requires $q = \frac{Uh}{2} + \frac{h^3}{12\eta}\frac{p_a - 0}{a}$ — ①

$$q = \frac{Uh_p}{2} + \frac{h_p^3}{12\eta}\frac{p_2 - 0}{b} \quad — ②$$

and $$q = \frac{Uh}{2} + \frac{h^3}{12\eta}\,\frac{p_2 - p_a}{c} \quad — \quad (3)$$

Eliminating p_2 from these equations

eq. from (1) & (3) $\quad p_2 = \left(1 + \frac{c}{a}\right) p_a$

thus from (2)

$$\frac{6\eta U}{p_a}\left(h_p - h\right) = \left(1 + \frac{c}{a}\right)\frac{h_p^3}{b} + \frac{h^3}{a}$$

(iv) Load carrying capacity = shaded area $- p_a(a+b+c)$

(many candidates forgot this term)

i.e. $$\frac{W}{L} = \frac{1}{2}p_a a + \frac{1}{2}p_2 b + \frac{1}{2}(p_2 - p_a)c + c p_a - p_a(a+b+c)$$

hence $$\frac{2W}{p_a L} = a + b + \frac{bc}{a} + \frac{c^2}{a} - 2a - 2b$$

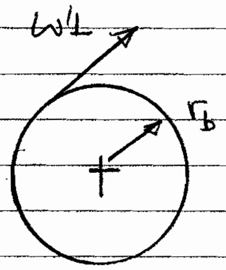
i.e. $$\frac{W}{L} = \frac{p_a c}{2}\left\{\frac{b+c}{a} - \frac{a+b}{c}\right\}$$

May be part of the explanation of load carrying capacity of laser textured surfaces – see Etsion, I 'Improving tribological performance of mechanical components' Trib. Letters 17 (2004) 733-737.

Examiner's comments Generally well done – average mark 14.1/20.

2 (a) For Kapitza and Ertel-Grubin see any of standard texts. Two important effects in EHL are elastic flattening of the non-conforming profiles and local increase in viscosity of lubricant. As the load on such a contact is increased so the length of the essentially central parallel region extends but the thickness of the lubricant film decreases only slightly. The thickness of a EHL film is much more dependent on velocity than load.

(b) (i)



Contact stress

$$p_0 = \left(\frac{W'E^*}{\pi R}\right)^{1/2} \qquad W' = \frac{W}{\ell}$$

Torque $T = r_b W$

For scaling of all dimensions the radius of curvature R at the critical location will scale directly with size ie. $\propto \alpha$.

ie. $r_b \propto \alpha$; $\ell \propto \alpha$ while E^* fixed

Need to match p_0 in both gear sets to get same fatigue failure condition

1 = original
2 = revised

$$\frac{T_1}{T_2} = \frac{r_{b1} W_1}{r_{b2} W_2}$$

ie. $$\frac{1}{5} = \frac{r_{b1}}{r_{b2}} \cdot \frac{W_1'}{W_2'} \cdot \frac{L_1}{L_2} \Rightarrow \frac{r_{b1}}{r_{b2}} \cdot \frac{R_1}{R_2} \cdot \frac{L_1}{L_2} = \frac{1}{\alpha^3}$$

$$\therefore \alpha = \sqrt[3]{5} = 1.71$$

Thus preferred module $= 5 \times 1.71 = 8.54$ mm

Best to go up to 9.0 mm ie. $\alpha = 1.8$

(b)(iii) Now use bending formula (data sheet)

$$\sigma_b = \frac{P_T'}{Jm} \qquad P_T' = \frac{T}{r_b \ell}$$

$\therefore$ If $\sigma_{b_2} = \sigma_{b_1}$

$$1 = \frac{T_2}{T_1} \frac{r_{b_1}}{r_{b_2}} \frac{\ell_1}{\ell_2} \cdot \frac{J_1}{J_2} \frac{m_1}{m_2}$$

But $\ell_2 = \ell_1$ and $m_2 = m_1$

$\therefore$ if $\frac{T_2}{T_1} = 5$, $5 = \frac{J_2}{J_1} \cdot \frac{r_{b_2}}{r_{b_1}}$

Now from Fig on p4 of Data sheet $J_1 \simeq 0.29$

and since m is const. $r_b \propto n$ no of teeth

$$\therefore 5 = \frac{J_2}{0.29} \cdot \frac{n_2}{17}$$

Trial & error, try $n_2 = 50$, $J_2 \simeq 0.44$

So RHS $= \frac{0.44}{0.29} \frac{50}{17} = 4.46$ not enough

try $n_2 = 80$ $J_2 \simeq 0.48$

$$\frac{0.48}{0.29} \frac{80}{17} = 7.79 \quad \text{too much}$$

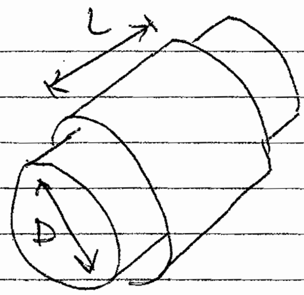
try interpolating linearly $\frac{n_2 - 50}{5 - 4.46} = \frac{80 - 50}{7.79 - 4.46}$

hence $n_2 \simeq 54.9$

So put $n_2 = 55$ and matching gear 165

Examiner's comments: not a popular question, average mark only 8.81/20.

3 (a)



In a 'long' bearing axial flow of lubricant is negligible, ~~w~~ which means that axial pressure gradients are small c/f to those around circumference.

In 'short' bearings axial pressure gradients are dominant. Bearings with $L/D > 4$ can be considered long.

(b) Given eqn.

$$W = \frac{\pi}{4} \frac{\eta \omega \varepsilon R L^3}{c^2(1-\varepsilon^2)^2} \left\{ \left(\frac{16}{\pi^2} - 1 \right) \varepsilon^2 + 1 \right\}^{1/2}$$

i.e. $$\frac{4}{\pi} \frac{W}{\eta \omega R L^3} = \frac{\varepsilon}{c^2(1-\varepsilon^2)^2} \left\{ \left(\frac{16}{\pi^2} - 1 \right) \varepsilon^2 + 1 \right\}^{1/2}$$

Now put $\varepsilon = 1 - \delta$

and noting that $h_{min} = (1-\varepsilon)c = \delta c$

then $(1-\varepsilon^2) = 1 - (1-\delta)^2 = 1 - 1 + 2\delta \ldots \simeq 2\delta$

$$\varepsilon^2 = (1-\delta)^2 \simeq 1 - 2\delta$$

then $$\frac{4}{\pi} \frac{W}{\eta \omega R L^3} = \frac{1-\delta}{c^2 (2\delta)^2} \left\{ \left(\frac{16}{\pi^2} - 1 \right)(1-2\delta) + 1 \right\}^{1/2}$$

$$= \frac{1-\delta}{4c^2\delta^2} \left\{ \frac{16}{\pi^2} - 1 + 1 \right\}^{1/2}$$

$$= \frac{\cancel{4}}{\cancel{4}\pi c^2 \delta^2}$$

$$= \frac{1}{\pi h_{min}^2}$$

$$h_{min} \simeq \frac{1}{2} \sqrt{\frac{\eta \omega R L^3}{W}} \qquad \text{QED}$$

(c)

Friction power loss

$$P \Rightarrow \frac{\pi \eta \omega^2 L R^3 (2+\varepsilon)}{c(1+\varepsilon)\sqrt{1-\varepsilon^2}}$$

clearly when $\varepsilon \to 0$ Petrov

(d) $$\frac{P \, . \, c}{\pi \eta \omega^2 L R^3} = \frac{2+\varepsilon}{(1+\varepsilon)(1-\varepsilon^2)^{1/2}}$$

again putting $\varepsilon = 1-\delta$

$$\text{RHS} = \frac{2+(1-\delta)}{(2-\delta)\sqrt{2\delta}}$$

$$\frac{Pc}{\pi \eta \omega^2 L R^3} \simeq \frac{3}{2\sqrt{2\delta}}$$

But $$\frac{4W}{\eta \omega R L^3} \simeq \frac{1}{c^2 \delta^2}$$

i.e. $$\delta = \left\{\frac{\eta \omega R L^3}{4 W c^2}\right\}^{1/2}$$

$$\therefore \frac{Pc}{\pi \eta \omega^2 L R^3} \simeq \frac{3}{2\sqrt{2}} \left(\frac{4Wc^2}{\eta \omega R L^3}\right)^{1/4}$$

$$P \simeq \frac{3\pi \sqrt{2}}{2\sqrt{2}} \frac{\eta \omega^2 L R^3}{c} \cdot \frac{W^{1/4} c^{1/2}}{\eta^{1/4} \omega^{1/4} R^{1/4} L^{3/4}}$$

$$= \left(\frac{3\pi}{2}\right) \eta^{3/4} \frac{\omega^{1.75} L^{0.25} R^{2.75} W^{0.25}}{c^{1/2}}$$

Examiner's comments: average mark 13.8/20 many candidates got stuck in part (d) though some complete solutions.

Petrov $P_\mu = 2\pi \eta \omega^2 L R^3 / c$

Full $P'_\mu = \frac{3\pi}{2} \eta^{3/4} \frac{\omega^{1.75} L^{0.25} R^{2.75} W^{0.25}}{c^{0.5}}$

$\therefore$ If $R = 0.02\text{m}$; $L = .02$; $\eta = .03$ Pas $\omega = 60\text{s}^{-1}$

$W = 7.2 \times 10^3$ $c/R = .001$ ie. $c = .02 \times 10^{-3}$ m

Petrov $P_\mu = \frac{2\pi \times .03 \times 60^2 \times .02 \times .02^3}{.02 \times 10^{-3}} = \underline{5.43\text{ W}}$

$$P'_\mu = \frac{3\pi}{2} \times \frac{.03^{0.75}\ 60^{1.75}\ .02^{.25}\ .02^{2.75}\ 7200^{.25}}{(.02 \times 10^{-3})^{.5}}$$

$= \underline{7.24\text{ Watts}}$

ie. $\frac{P_\mu}{P'_\mu} = \underline{\underline{75\%}}$

Check $S = \eta\omega \frac{LD}{W}\left\{\frac{R}{c}\right\}^2 = .03 \times 60 \times \frac{.02 \times .04}{7200}\{1000\}^2$

$= 0.2$ ✓ ok

$$h_{min} = \frac{1}{2}\sqrt{\frac{.03 \times 60 \times .02 \times .02^3}{7200}} \Rightarrow \underline{3.16\,\mu m}$$

$c = 10^{-3} \times .02 = 20\,\mu m$ $\therefore$ $\varepsilon = \frac{20 - 3.16}{20}$

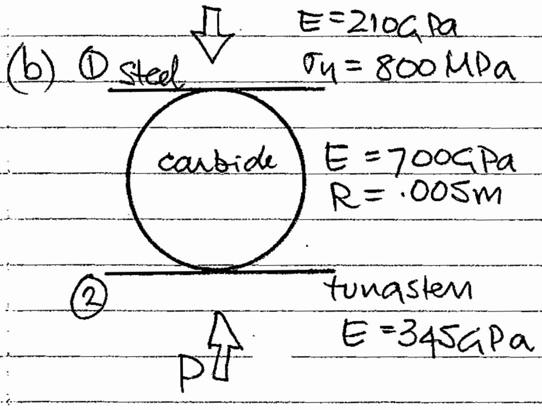
$= \underline{\underline{0.84}}$

Higher shear rates in narrow film leads to increased local surface shear stresses.

For discussion see R.I. Taylor 'Simplifications to the short bearing approximation' Proc Instn Mech Engrs 218 J (2004) 569-573

4(a) Bookwork – but essentially: small strains, i.e. $\delta \ll R$; linear elasticity, no friction (or identically mat'ls); model circular profiles as parabolae

(b)



$$E_1^* = \left[\frac{1-.3^2}{210} + \frac{1-.3^2}{700}\right]^{-1} \Rightarrow 178\,\text{GPa}$$

$$E_2^* = \left[\frac{1-.3^2}{345} + \frac{1-.3^2}{700}\right]^{-1} \Rightarrow 254\,\text{GPa}$$

(i) Treat both contacts as Hertzian

Data Sheet $\quad \left(\frac{a_1}{a_2}\right)^3 = \frac{E_2^*}{E_1^*} \quad \therefore \frac{A_1}{A_2} = \left(\frac{a_1}{a_2}\right)^2 = \left(\frac{254}{178}\right)^{2/3} = 1.27$

$$\delta = \frac{1}{2}\left(\frac{9P^2}{2E^{*2}R}\right)^{1/3} \quad \text{Data Sheet}$$

$$\text{So} \quad \frac{\delta_1}{\delta_2} = \left(\frac{E_2^*}{E_1^*}\right)^{2/3} = 1.27$$

But $\delta_1 + \delta_2 = 0.001\,\text{mm} = 1\,\mu\text{m}$

$$\therefore \delta_1 + \frac{\delta_1}{1.27} = 1 \quad \therefore \begin{cases} \delta_1 \Rightarrow 0.56\,\mu\text{m} \\ \delta_2 \Rightarrow 0.44\,\mu\text{m} \end{cases}$$

thus P given by $\frac{9P^2}{2E^{*2}R} = 8\delta^3$

$$\therefore P = \left\{\frac{8 \times (0.56\times10^{-6})^3 \times 2 \times (178\times10^9)^2 \times .005}{9}\right\}^{1/2}$$

i.e. $P \Rightarrow 7.03\,\text{N}$

To confirm elasticity calculate τ_{max} and c/f to given values.

$$p_{o_1} = \frac{1}{\pi}\left(\frac{6PE^{*2}}{R^2}\right)^{1/3}$$ Data Sheet

$$\therefore p_{o_1} = \frac{1}{\pi}\left(\frac{6 \times 7.03 \times (178\times10^9)^2}{.005^2}\right)^{1/3} \Rightarrow 1.20\,GPa$$

But $\tau_{1max} \simeq 0.31\,p_o \simeq 371\,MPa$

Data Sheet

Since this less than $\tau_y \left(= \frac{\sigma_y}{2}\right)$ i.e. 400 MPa

elasticity OK

at other contact $\frac{p_{o_2}}{p_{o_1}} = \left(\frac{E_2^*}{E_1^*}\right)^{2/3}$ i.e. $p_{o_2} = \left(\frac{254}{178}\right)^{2/3} \times 1.2$

$= 1.52\,GPa$

$\therefore \tau_{max_2} = .31 \times 1.52 = 470\,MPa.$

Since $\frac{470}{400} < 1.4$ contact ② also elastic

(ii) when limit of elasticity reached at steel plate

$\tau_{max_1} = 0.31\,p_o = 400\,MPa \quad \therefore p_{o_1} = 1.29\,GPa$

$$\therefore P = 7.03 \times \left(\frac{1.29}{1.20}\right)^3 \Rightarrow \underline{8.74\,N}$$

τ_{max_2} will now be $\left(\frac{8.75}{7.03}\right)^{1/3} \times 470 = 505\,MPa$

Since this is still less than 1.4×400 conditions are still elastic.

when $P = 8.74N$ $\quad \delta_1 = \left(\frac{8.75}{7.03}\right)^{2/3} \times 0.56 \Rightarrow 0.65\,\mu m$

and $\delta_2 = \frac{0.65}{1.27} = 0.51\,\mu m$

$$\therefore \delta = 0.65 + 0.51 \Rightarrow \underline{1.16\,\mu m}$$

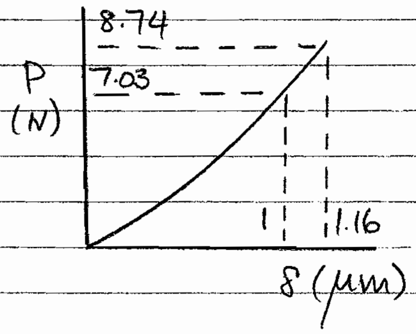
While contact is elastic $\quad \delta = \delta_1 + \delta_2$

i.e. $\delta = \frac{1}{2}\left(\frac{9P^2}{2R}\right)^{1/3}\left\{\frac{1}{E_1^{*2/3}} + \frac{1}{E_2^{*2/3}}\right\}$

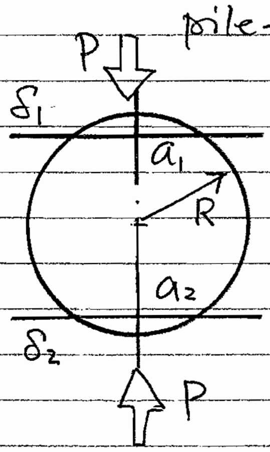
i.e. $\delta = \frac{1}{2}\left(\frac{9}{.07}\right)^{1/3}\left\{\frac{1}{(178\times10^9)^{2/3}} + \frac{1}{(254\times10^9)^{2/3}}\right\} P^{2/3}$

$\delta \Rightarrow 2.73\times10^{-7} P^{2/3}$ or $P = 3.67\times10^6 . \delta^{3/2}$

[As a check put $P = 7.03$N, then $\delta = 1.00\times10^{-6}$m ✓]



(iii) When $\delta = 0.15$ mm, full plasticity, neglecting pile-up



$\delta . 2R \simeq a^2 \quad \therefore \delta_1 \simeq \frac{a_1^2}{2R}; \quad \delta_2 \simeq \frac{a_2^2}{2R}$

$\frac{P}{\pi a^2} \simeq 3Y \simeq 6\tau_y$

$\tau_{y_1} \simeq 400$MPa $\quad \therefore \frac{\tau_{y_2}}{\tau_{y_1}} = \left(\frac{a_1}{a_2}\right)^2 = \frac{\delta_1}{\delta_2}$

$= \left(\frac{1}{.75}\right)^2 = 1.78$

$\therefore \tau_{y_2} \sim 1.78\times400$

$\Rightarrow 711$ MPa

$P = \pi a^2 . 6\tau_y \Rightarrow \pi\times(.001)^2\times6\times400\times10^6$

$= 7.54$ kN

(Brinell test uses 3000 kgf i.e. ~ 30kN)

Examiner's comments: average mark 11.1/20, usually arithmetic errors.

JAW/May 07