PART IA ENGINEERING MODULE 3C6 - 2007

1. (a) No mass. "kinetic energy" $\tilde{T} = \frac{1}{2}\frac{M}{L}\int u^2 dx$

$$= \frac{1}{2}\frac{M}{L}\int_0^L \sin^2\frac{n\pi x}{L}dx = \frac{M}{2L}\cdot\frac{L}{2} = M/4$$

Potential energy $V = \frac{1}{2}P\int u'^2 dx$

$$= \frac{1}{2}P\int_0^L\left(\frac{n\pi}{L}\right)^2\cos^2\frac{n\pi x}{L}dx = \frac{P}{2}\left(\frac{n\pi}{L}\right)^2\cdot\frac{L}{2} = \frac{Pn^2\pi^2}{4L}$$

So by Rayleigh, $\Omega_n^2 = \frac{V}{\tilde{T}} = \frac{Pn^2\pi^2}{ML}$

(exact because mode shape is exact)

(b) With added mass: potential energy is unchanged.
New ~~kinetic~~ energy $\tilde{T} = \frac{M}{4} + \frac{1}{2}m u_n^2(X) = \frac{M}{4} + \frac{m}{2}\sin^2\left(\frac{n\pi X}{L}\right)$

So $\omega_n^2 = \dfrac{Pn^2\pi^2/4L}{M/4 + \frac{m}{2}\sin^2\left(\frac{n\pi X}{L}\right)} = \Omega_n^2\left[1 + \frac{2m}{M}\sin^2\left(\frac{n\pi X}{L}\right)\right]^{-1}$

$\therefore\ \omega_n \simeq \Omega_n\left[1 - \frac{m}{M}\sin^2\left(\frac{n\pi X}{L}\right)\right]$ by binomial.

(c) Mass has biggest effect when placed at an antinode
So that $\sin^2\frac{n\pi X}{L} = 1$

Then $\frac{\omega_n}{\Omega_n} = 1 - \frac{m}{M}$

For one semitone drop, want $\frac{\omega_n}{\Omega_n} = 2^{-1/12} = 0.944$

So want $m = 0.056\,M$

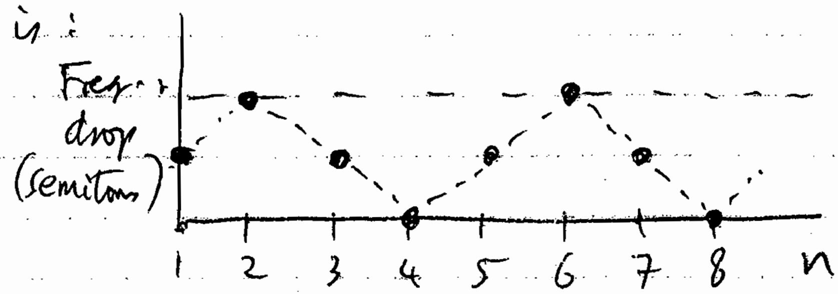
For second mode, place mass at $L/4$ or $3L/4$.

This a node for modes $n = 4, 8, 12 \ldots$ so has no effect on these modes. It is an antinode for $n = 2, 6, 10 \ldots$
Effect on modes $n = 1, 3, 5, 7 \ldots$ is equal: $1 - \frac{m}{M}\sin^2\frac{\pi}{4} = 1 - \frac{m}{2M}$

1 (cont)

So $\omega_n = \Omega_n\left(1 - \frac{0.056}{2}\right)$, which is $\frac{1}{2}$ semitone approximately

So graph is:



(d) For a beam with same mass per unit length, the kinetic energy is the <u>same</u> if the mode shape is the same — OK for pinned-pinned beam.

Now potential energy $V_{beam} = \frac{1}{2} EI \int u''^2 dx$

So without mass: $\Omega_n^2 = \dfrac{V}{M/4}$

with mass: $\omega_n^2 \simeq \dfrac{V}{\frac{M}{4} + \frac{m}{2}\sin^2\frac{n\pi x}{L}}$

$\therefore\ \omega_n \simeq \Omega_n\left[1 - \frac{m}{M}\sin^2\frac{n\pi x}{L}\right]$ exactly as before.

For different boundary conditions, need to use the relevant mode shapes $u_n(x)$ for the unloaded beam in place of $\sin\frac{n\pi x}{L}$.

Note that for the pinned-pinned case the mass-normalised mode shape is $\sqrt{\frac{2}{M}}\sin\frac{n\pi x}{L}$

So if $u_n(x)$ is relevant mass-normalised shape,

$$\omega_n \simeq \Omega_n\left[1 - \frac{m}{2}u_n^2(x)\right]$$

2 (a) The system is mirror-symmetric about $x=0$, so the modes are either symmetric or antisymmetric. The free solutions of the equation of motion are
$$u = e^{\pm\alpha x},\ e^{\pm i\alpha x} \quad \text{where } \alpha^4 = \frac{m\omega^2}{EI} \quad (1)$$

So for symmetric modes, must have $u = A\cos\alpha x + B\cosh\alpha x$, for antisymmetric modes $u = A\sin\alpha x + B\sinh\alpha x$

At $x=L$, no motion so $u=0$.
Second boundary condition states that bending moment must balance the moment from the torsion spring, so $EI\frac{\partial^2 u}{\partial x^2} = -K\frac{\partial u}{\partial x}$

or $\frac{\partial^2 u}{\partial x^2} = -\lambda\frac{\partial u}{\partial x}$ with $\lambda = \frac{K}{EI}$

(b) For symmetric modes choose $u = A\cos\alpha x + B\cosh\alpha x$

$$\text{At } x=L: \begin{cases} A\cos\alpha L + B\cosh\alpha L = 0 \\ \alpha^2[-A\cos\alpha L + B\cosh\alpha L] = -\lambda\alpha[-A\sin\alpha L + B\sinh\alpha L] \end{cases}$$

For non-trivial solutions $\begin{vmatrix} \cos\alpha L & \cosh\alpha L \\ -\lambda\sin\alpha L - \alpha\cos\alpha L & \lambda\sinh\alpha L + \alpha\cosh\alpha L \end{vmatrix} = 0$

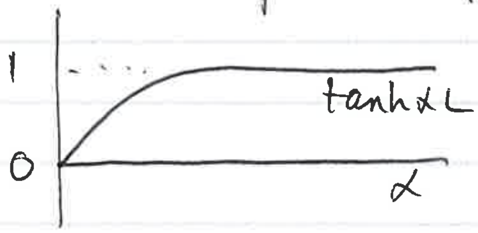
$\therefore \lambda\cos\alpha L\sinh\alpha L + \alpha\cos\alpha L\cosh\alpha L + \lambda\sin\alpha L\cosh\alpha L + \alpha\cos\alpha L\cosh\alpha L$

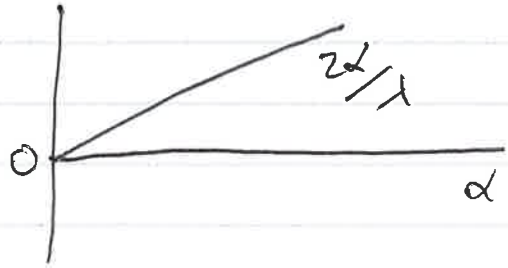
$$\therefore \lambda\tanh\alpha L + \lambda\tan\alpha L + 2\alpha = 0$$
$$\text{or} \quad \tanh\alpha L + \tan\alpha L + 2\alpha/\lambda = 0$$

Roots for α determine ω from (1).

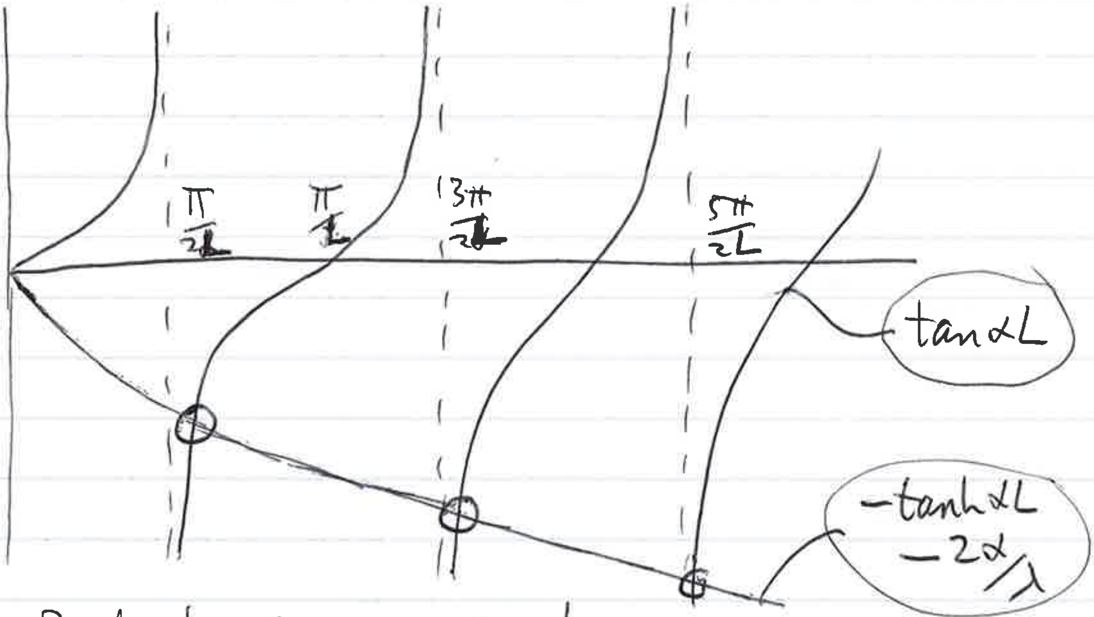
Rearrange as $\tan\alpha L = -\tanh\alpha L - \frac{2\alpha}{\lambda}$

(c) Then as a function of α:





So:



Roots for α are ringed.

As $\lambda \to 0$ (i.e. $K \to 0$, pinned-pinned beam)
Then roots tend to $\frac{1}{L}\left(n - \frac{1}{2}\right)\pi$

As $\lambda \to \infty$ (ie $K \to \infty$, clamped-clamped beam)
roots tend to solutions of $\tan \alpha L = -\tanh \alpha L$
So for higher frequencies, $\tan \alpha L \simeq -1$
ie $\alpha \simeq \frac{1}{L}\left(n - \frac{1}{4}\right)\pi$

(d) No real clamp is infinitely rigid, either in rotation or lateral stiffness. A real "clamped" beam will always show results intermediate between the ideal clamped and pinned values. High modes tend more towards the "pinned" case: provided λ is finite, the term $-2\alpha/\lambda$ always dominates eventually.

3.(a) $k = \frac{3EA}{L}$, $m = M/4$

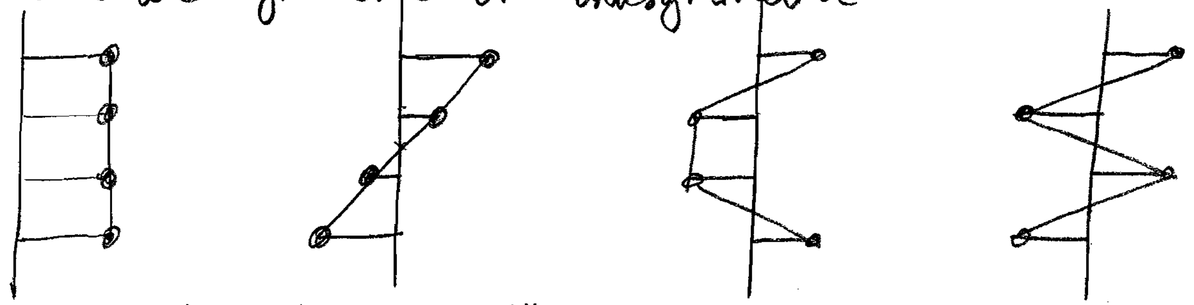
Mass matrix by inspection $[M] = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{bmatrix}$

Stiffness matrix by energy:

$$V = \tfrac{1}{2}k\left[(y_2-y_1)^2 + (y_3-y_2)^2 + (y_4-y_3)^2\right]$$
$$= \tfrac{1}{2}k\left[y_1^2 + 2y_2^2 + 2y_3^2 + y_4^2 - 2y_1y_2 - 2y_2y_3 - 2y_3y_4\right]$$
$$= \tfrac{1}{2}\,\underset{\sim}{q}^T [k]\, \underset{\sim}{q}$$

where $[k] = k\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

(b) Modes are symmetric or antisymmetric

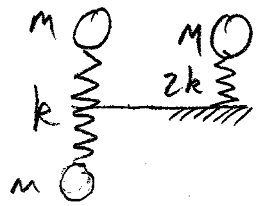


(c) guessed mode shapes $\underset{\sim}{u}^{(j)}$

$$\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \quad \begin{Bmatrix} 3 \\ 1 \\ -1 \\ -3 \end{Bmatrix} \quad \begin{Bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{Bmatrix} \quad \begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix}$$

Mode 1 : rigid body translation $\Rightarrow \omega_1^2 = 0$ <u>Exact</u>

Mode 3 : Spring between masses 2 & 3 is unstretched, so masses 1 & 2 vibrate independently of masses 3 & 4. Therefore the frequency is as if each pair vibrates with a node at mid-spring



$\Rightarrow \omega_3^2 = 2k/m$ <u>Exact</u>

3 cont.

Use Rayleigh for modes 2 & 4, with $T = \frac{1}{2}m(y_1^2 + y_2^2 + y_3^2 + y_4^2)$

<u>Mode 2</u> $\underset{\sim}{u}^{(2)} = [3 \quad 1 \quad -1 \quad -3]^T$

$$\omega_2^2 \simeq \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k\left[(1-3)^2 + (-1-1)^2 + (-3-(-1))^2\right]}{\frac{1}{2}m\left[3^2 + 1^2 + (-1)^2 + (-3)^2\right]} = \frac{k}{m}\frac{12}{20} = \underline{\underline{0.6\frac{k}{m}}}$$

<u>approximate</u>

Mode 4 $u^{(4)} = [1 \quad -1 \quad 1 \quad -1]^T$

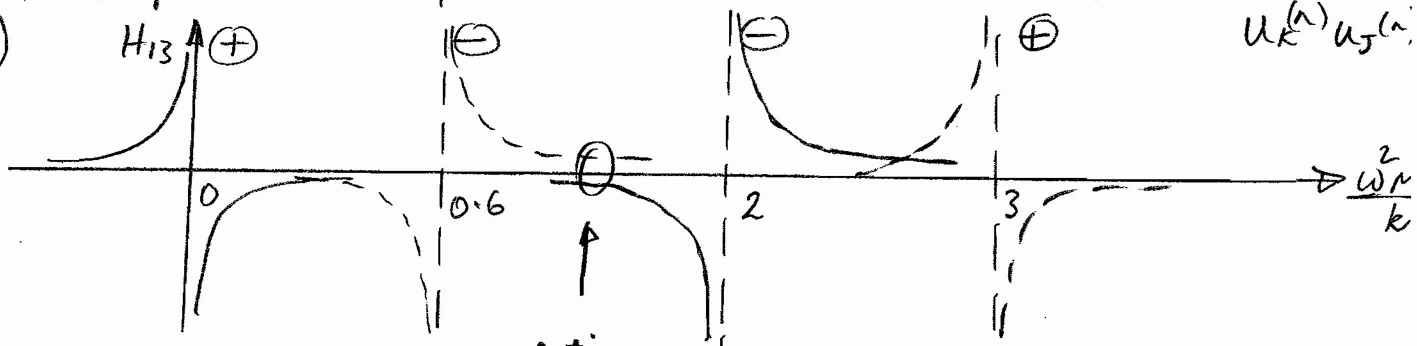
$$\omega_4^2 \approx \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k\left[(-1-1)^2 + (1-(-1))^2 + (-1-1)^2\right]}{\frac{1}{2}m\left[1^2 + (-1)^2 + (1)^2 + (-1)^2\right]} = \frac{k}{m}\frac{12}{4} = \underline{\underline{3\frac{k}{m}}}$$

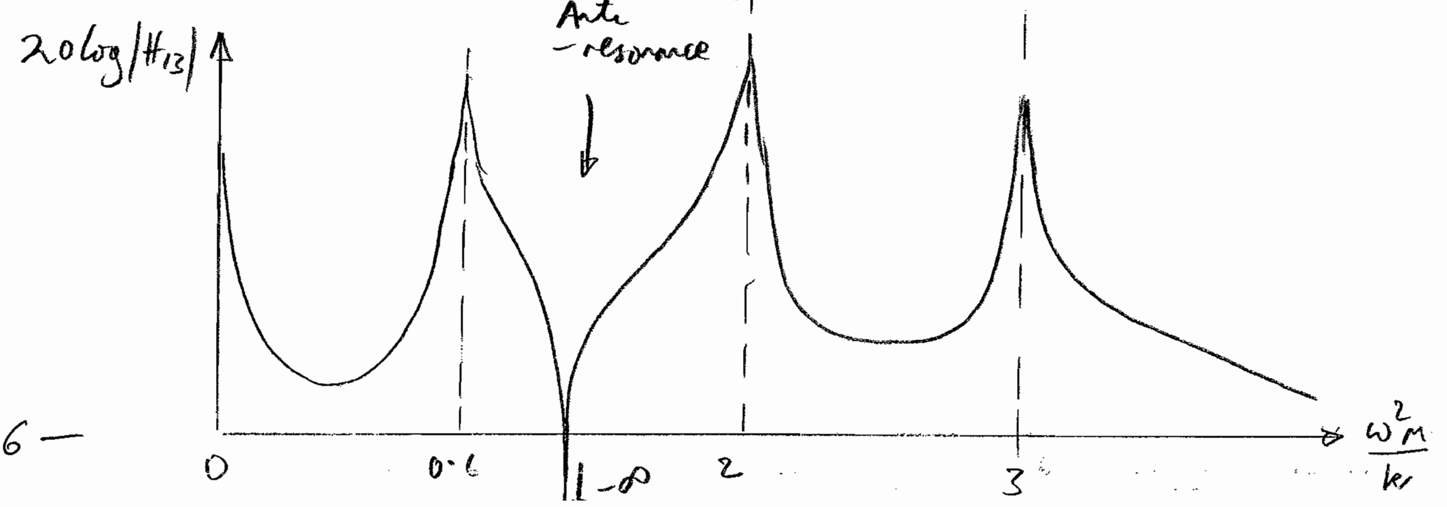
<u>approximate</u>

(c) Mode 2 must be symmetric, even though the guessed shape is not exact. So in mode 2, $y_1 = -y_4$.

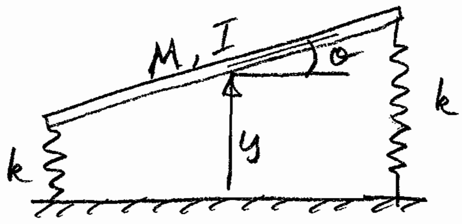
So when the column is excited in mode 2, the response at the top must be of same magnitude as the base, but 180° out of phase. (NB No damping is needed to maintain a finite response for a displacement input.

(d)





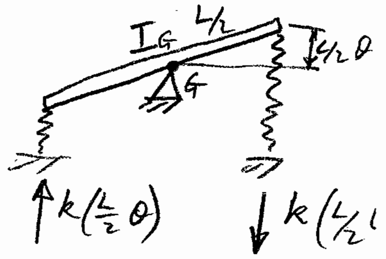
4. (a)



(i) Symmetric mode: $\theta = 0$ "bounce"

$\underline{\omega_1^2 = 2k/m}$

(ii) Anti-symmetric mode: $y = 0$ "pitch"

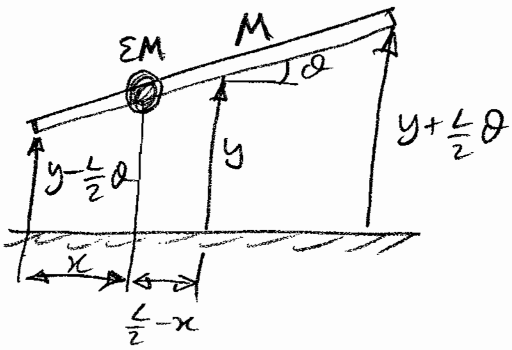


$\Sigma M_G: \quad 2\left(\frac{L}{2}\right)k\left(\frac{L}{2}\theta\right) = I_G \ddot{\theta}$

with $I_G = \frac{1}{12}ML^2$

$$\Rightarrow \frac{kL^2}{2}\theta = \frac{ML^2}{12}\ddot{\theta} \Rightarrow \underline{\underline{\omega_2^2 = \frac{6k}{M}}}$$

(b) With a small added mass ($\varepsilon \ll 1$) it is reasonable to assume that the mode shapes don't change significant from the modes in Part (a). ∴ Use these modes in Rayleigh's quotient.



PE $\quad V = \frac{1}{2}k\left[\left(y+\frac{L}{2}\theta\right)^2 + \left(y-L/2\,\theta\right)^2\right]$

KE $\quad T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}(\varepsilon M)\left(\dot{y} - (L/2 - x)\dot{\theta}\right)^2$

Mode (i) with $\theta = 0$:

$$\omega_1^2 \approx \frac{V_{max}}{T^*} = \frac{\frac{1}{2}k\left[y^2 + y^2\right]}{\frac{1}{2}M\left[y^2 + \varepsilon(y^2)\right]} = \underline{\underline{\frac{k}{M}\frac{2}{(1+\varepsilon)}}}$$

Mode (ii) with $y = 0$:

$$\omega_2^2 \approx \frac{\frac{1}{2}k\left[L^2/4\,\theta^2 + L^2/4\,\theta^2\right]}{\frac{1}{2}\left(\frac{1}{12}ML^2\right)\left[\theta^2\right] + \frac{1}{2}\varepsilon M\left(\frac{L}{2}-x\right)^2\theta^2} = \frac{k}{M}\frac{L^2/2}{1/12\,L^2 + \varepsilon\left(\frac{L}{2}-x\right.}$$

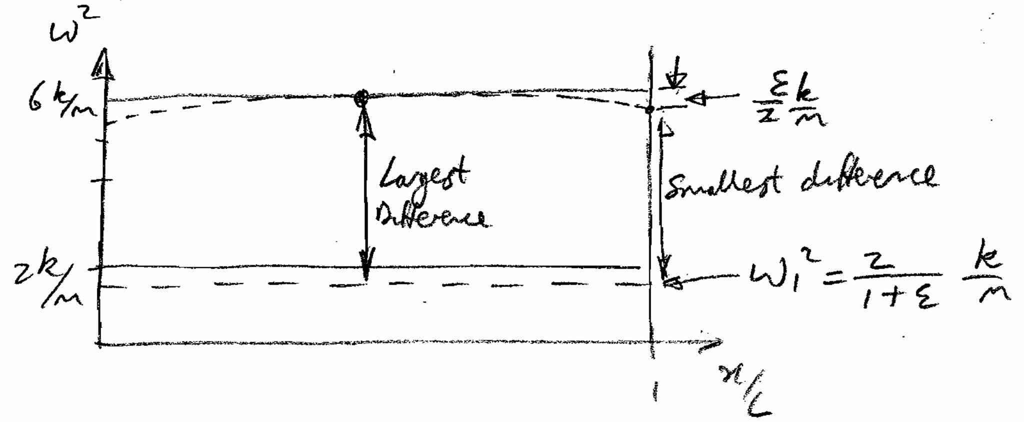
$$= k/m \cdot \frac{1}{1/6 + 2\varepsilon\left(\frac{1}{2}-x/L\right)^2} \approx \underbrace{\frac{6k}{m}}_{\text{old freq.}} \underbrace{- \frac{2k}{m}\varepsilon\left(\frac{1}{2}-x/L\right)^2}_{\text{shift due to } \varepsilon M}$$

(using binomial expansion)

4(c) Shift in lower mode is independent of x, so the size of the difference in frequency between the two modes depends only on the value of ω_2^2.

(i) Largest difference is when ω_2^2 is highest, which happens when $x = L/2$ ie additional mass is in the middle. In this case ω_2^2 is unchanged from the value in part (a)

(ii) The smallest difference is when ω_2^2 is lowest, which happens when $x=0$ or $x=L$. At these positions the pitch moment of inertia of the system is largest.



ENGINEERING TRIPOS PART IIA

Module 3C6 Examination, 2007

Answers

1 (a) $\Omega_n^2 = \dfrac{Pn^2\pi^2}{ML}$ (c) $m = 0.056\,M$ at L/4 or 3L/4 (d) Same as (b)

2 (a)(i) $\alpha^4 = \dfrac{m\omega^2}{EI}$ (a)(ii) $\lambda = \dfrac{K}{EI}$

3 (a) $k = \dfrac{3EA}{L}$, $m = \dfrac{M}{4}$, $\mathbf{M} = m\mathbf{I}$ (4x4); $\mathbf{K} = k\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

(c) $\omega_1^2 = 0$ (exact); $\omega_2^2 = 0.6\dfrac{k}{m}$; $\omega_3^2 = 2\dfrac{k}{m}$ (exact); $\omega_4^2 = 3\dfrac{k}{m}$

(e) ±1mm at 180° out of phase with excitation

4 (a) Pure 'bounce' $\omega_1^2 = 2\dfrac{k}{m}$; Pure 'pitch' $\omega_2^2 = 6\dfrac{k}{m}$

(b) $\omega_2^2 \approx 6\dfrac{k}{m} - \dfrac{2k}{m}\left(\dfrac{1}{2} - \dfrac{x}{L}\right)^2 \varepsilon$

(c) largest difference when $x = L/2$; smallest difference when $x = 0$ or $x = L$.