PART IIA 2007
3C7 Mechanics of solids
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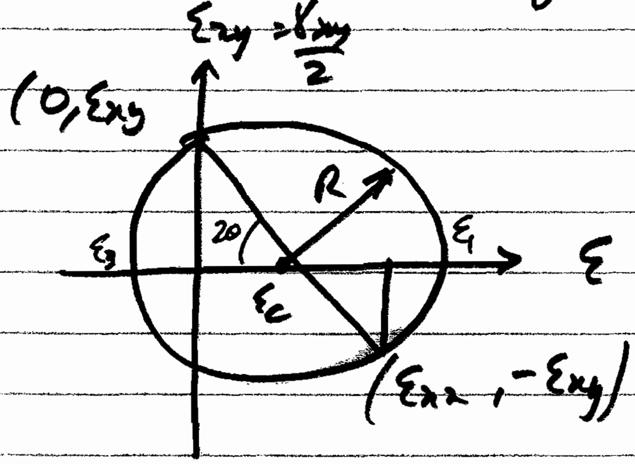
IIA 3C7 Solid Mechanics 2007
Civil.

1. (a) $\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\lambda y}{L^2}$ $\qquad \varepsilon_{yy} = 0$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right) = \frac{1}{2}\frac{\lambda}{L^2}\cdot x$$

0



$$\varepsilon_c = \frac{\varepsilon_{xx}}{2} = \frac{\lambda y}{2L^2}$$

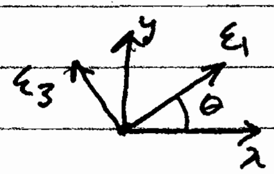
$$R^2 = \varepsilon_c^2 + \varepsilon_{xy}^2$$

$$\Rightarrow R = \frac{\lambda}{2L^2}\sqrt{x^2+y^2} = \varepsilon_{xy\,max}$$

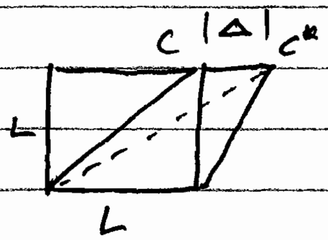
$$\varepsilon_1 = \varepsilon_c + R = \frac{\lambda}{2L^2}\left(y + \sqrt{x^2+y^2}\right) \qquad \varepsilon_2 = 0$$

$$\varepsilon_3 = \varepsilon_c - R = -\frac{\lambda}{2L^2}\left(\sqrt{x^2+y^2} - y\right)$$

$$\tan 2\theta = \frac{\varepsilon_{xy}}{\varepsilon_c} = \frac{x}{y}$$



(b)



$$\Delta = u(L,L) = \lambda$$

$$AC^* = L\sqrt{1 + \left(1 + \frac{2\lambda}{L} + \frac{\lambda^2}{L^2}\right)}$$

$$\approx \sqrt{2}L\left(1 + \frac{\lambda}{2L}\right)$$

$$\therefore\ \bar{\varepsilon}_{AC} = \frac{AC^* - AC}{AC} = \frac{\cancel{\sqrt{2}L}\left(\cancel{1} + \frac{1}{2}\frac{\lambda}{L}\right) - \cancel{\sqrt{2}L}}{\cancel{\sqrt{2}L}} = \frac{\lambda}{2L}$$

1 cont.

(c) Plane strain

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}) \quad (1)$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) = 0 \quad \text{(defined = Qu)} \quad (2)$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = 0 \quad \text{(plane strain)} \quad (3)$$

$$\gamma_{xy} = \frac{\sigma_{xy}}{G}.$$

From (3) $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ (4)

Sub this into (2) $\sigma_{yy} - \nu\sigma_{xx} - \nu^2\sigma_{xx} \cancel{+} - \nu^2\sigma_{yy} = 0$

$$\Rightarrow \sigma_{yy} = \frac{\nu}{(1-\nu)}\sigma_{xx} \quad (5)$$

Put (4) & (5) into (1) $E\varepsilon_{xx} = \sigma_{xx} - \frac{\nu^2\sigma_{xx}}{(1-\nu)} - \nu^2\sigma_{xx} - \frac{\nu^3\sigma_{xx}}{(1-\nu)}$

$\cancel{E\varepsilon_{xx} = \sigma_{xx} - \frac{\nu^2\sigma_{xx}}{(1-\nu)}}$

$$E\varepsilon_{xx}(1-\nu) = \sigma_{xx}\left[(1-\nu) - \nu^2 - \nu^2 + \cancel{\nu^3} - \cancel{\nu^3}\right]$$

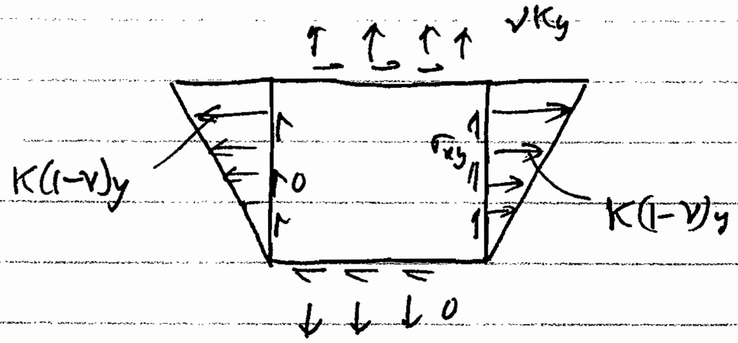
$$= \sigma_{xx}\left[1 - \nu - 2\nu^2\right] = \sigma_{xx}\left[1+\nu\right]\left[1-2\nu\right]$$

$$\therefore \sigma_{xx} = \frac{\varepsilon_{xx}E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\cdot\frac{\lambda y}{L^2}$$

$$\underline{\sigma_{yy}} = \frac{\nu\sigma_{xx}}{(1-\nu)} = \frac{E\nu}{(1+\nu)(1-2\nu)}\cdot\frac{\lambda y}{L^2}$$

$$\underline{\sigma_{zz}} = \nu(\sigma_{xx} + \sigma_{yy}) = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{\lambda}{L^2}y$$

$$\sigma_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\cdot 2\varepsilon_{xy} = \frac{E}{(1+\nu)}\frac{\lambda x}{2L^2}.$$



$$K = \frac{\lambda E}{(1+\nu)(1-2\nu)L^2}$$

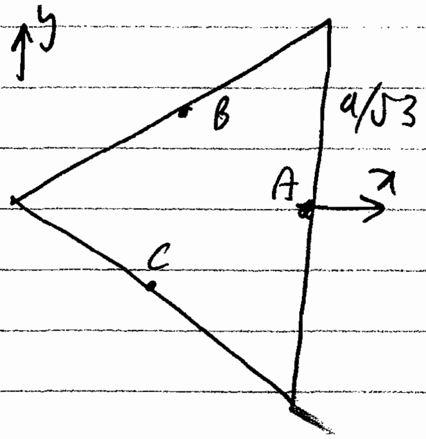
Equilibrium not satisfied in x, y or moment directions.

(d) $\tau_{max} = G\,\gamma_{max} = \frac{G\lambda}{L^2}\sqrt{x^2+y^2} = k$ for Tresca

First yield will occur where x^2+y^2 = maximum $= 2L^2$

$\therefore \quad \frac{G\lambda\sqrt{2}\cancel{L}}{L^{\cancel{2}}} = k \quad \Rightarrow \lambda = \frac{kL}{\sqrt{2}G}$

2 (a)



$$\phi = -\frac{G\alpha}{2}\left(2y^2 - x^2\right)\left(1 - x/a\right)$$

To be Prandtl function, $\phi = 0$ on boundary

automatically satisfied on $x = 0$

Must also be satisfied along $y = \pm\frac{x}{\sqrt{3}}$

$$2\frac{x^2}{3} - x^2 = 0 \quad \Rightarrow \quad 2 = 3$$

$$\therefore \phi = -\frac{G\alpha}{2}\left(3y^2 - x^2\right)\left(1 - x/a\right) \qquad \text{Does } \nabla^2\phi = -2G\alpha\ ?$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{G\alpha}{2}\left(-2 + \cancel{\frac{6x}{a}} - \cancel{\frac{6x}{a}} - 6\right) = -2G\alpha \quad \checkmark$$

(b) Torque $T = 2\int\phi\, dA$

$$T = -\frac{4G\alpha}{2}\int_0^a\left[\int_0^{x/\sqrt{3}}\left(3y^2 - 3y^2\frac{x}{a} - x^2 + \frac{x^3}{a}\right)dy\right]dx$$

$$= \frac{2a^4}{30\sqrt{3}}G\alpha = \underline{\underline{\frac{\sqrt{3}}{45}a^4 G\alpha}}$$

(c) The shear stress will be maximum at points A B & C. By symmetry they will all be the same.

$$\sigma_{xy}(a,0) = -\frac{\partial\phi}{\partial x}(a,0) = \frac{G\alpha}{2}\left(-\frac{3y^2}{a} - 2x + \frac{3x^2}{a}\right) = \frac{G\alpha a}{2}$$

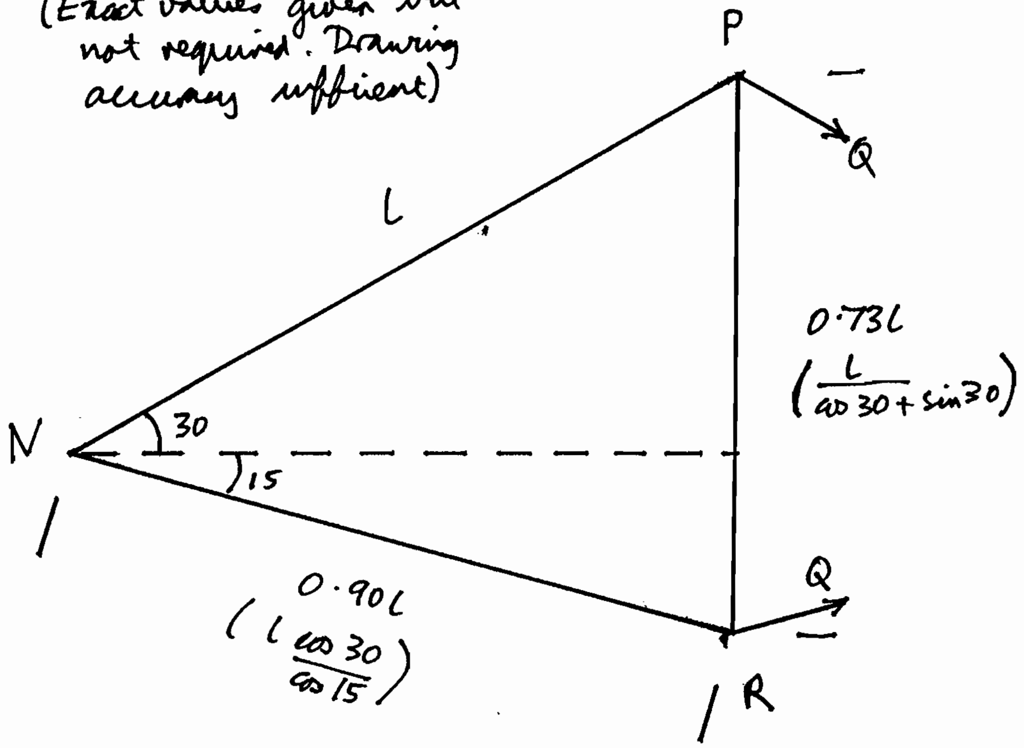
$$\sigma_{xy}(a,0) = \frac{45T}{2\sqrt{3}a^3} \quad \left(= \frac{15\sqrt{3}}{2}\frac{T}{a^3}\right)$$

(d) In corners and at centroid $\tau_{xy} = 0$.

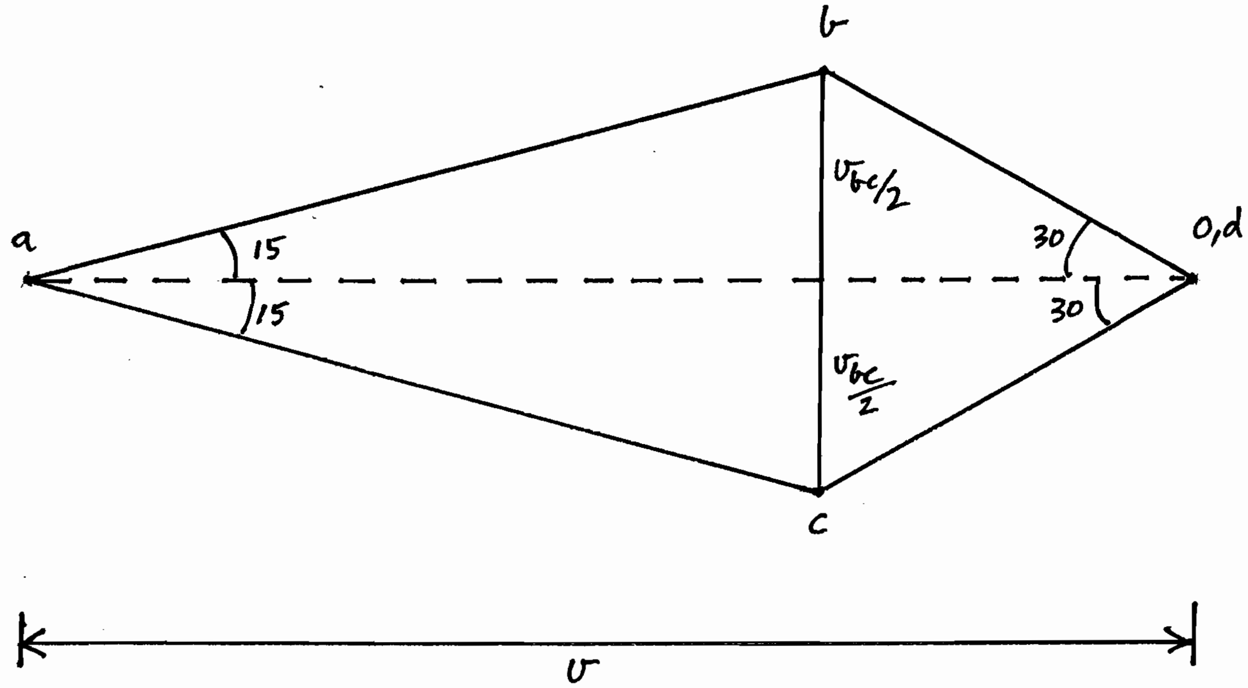
by symmetry

No shear stress
in each direction at edge
∴ Mohr's circle here must
vanish.

3(a) (Exact values given but not required. Drawing accuracy sufficient)



(b) velocity diagram



(c) From the velocity diagram

$$v_{bd} = \frac{v_{bc}}{2} \frac{1}{\sin 30}$$

$$v_{ab} = \frac{v_{bc}}{2} \cdot \frac{1}{\sin 15}$$

$$v = v_{ab} \cos 15 + v_{bd} \cos 30$$

$$= \frac{v_{bc}}{2} (\cot 15 + \cot 30)$$

$$= v_{bc} \, . \, 2.73$$

$\therefore v_{bc} = 0.366\, v \quad \Rightarrow \quad v_{bd} = v_{cd} = 0.366\, v$

$v_{ab} = 0.707 v = v_{ac}$

$$\therefore F v = K \left(v_{ab} \cdot l_{ab} + v_{bc} \cdot l_{bc} + v_{ac} \cdot l_{ac} \right)$$

$$= K \left(2 \times 0.90 L \, . \, 0.707 v + 0.73 L \, . \, 0.366 v \right)$$

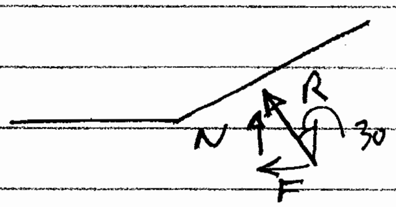
$$= K L v \, . \, 1.540$$

(Nothing from v_{cd} since stated to be frictionless)

$$\therefore \underline{\underline{F = 1.54\, KL}}$$

This will be an upper bound because it is based on a compatible mode of deformation and yield but does not consider equilibrium

(d)



R must be $\perp$ to interface

$$\therefore \frac{F}{N} = \tan 30 = 0.577$$

$$\therefore N = \frac{F}{0.577} = \frac{1.54\, KL}{0.577} = \underline{\underline{0.65\, KL}}$$

4 (a) $\phi = C_1 \sin 2\theta + C_2 \theta$

To be an Airy stress function, it must satisfy

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\underbrace{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}}_{\text{both zero}} + \underbrace{\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}}_{-4C_1 \sin 2\theta}\right) = 0$$

$$\frac{\partial^2}{\partial r^2}\left(-\frac{4C_1}{r^2}\sin 2\theta\right) = -24\frac{C_1}{r^4}\sin 2\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{4C_1}{r^2}\sin 2\theta\right) = \frac{8C_1}{r^4}\sin 2\theta$$

$$\frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\left(-\frac{4C_1}{r^2}\sin 2\theta\right) = \frac{16C_1}{r^4}\sin 2\theta$$

Sum $= 0$

$\therefore$ OK.

(b) (i) $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ on $\theta = \pm\alpha$

(ii) $$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = -\frac{4C_1}{r^2}\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = \frac{1}{r^2}\left(2C_1 \cos 2\theta + C_2\right)$$

$\theta = \alpha$ $\quad \sigma_{r\theta} = 0 \quad \therefore \quad 2C_1 \cos 2\alpha = -C_2$

$\theta = \alpha$ $\quad \sigma_{\theta\theta} = 0 \quad$ satisfied automatically

Generally $$\sigma_{rr} = -\frac{4C_1}{r^2}\sin 2\theta$$

$$\sigma_{r\theta} = \frac{2C_1}{r^2}\left(\cos 2\theta - \cos 2\alpha\right)$$

(c) $$M = \int_{-\alpha}^{\alpha} \sigma_{r\theta} \cdot r^2 = 2C_1 \left(\sin 2\alpha - 2\alpha \cos 2\alpha \right)$$

$$\Rightarrow C_1 = \frac{M}{2(\sin 2\alpha - 2\alpha \cos 2\alpha}$$

$$\sigma_{rr} = \frac{-2M \sin 2\theta}{r^2(\sin 2\alpha - 2\alpha \cos 2\alpha)}$$

$$\sigma_{r\theta} = \frac{M \; (\cos 2\theta - \cos 2\alpha)}{r^2 \; (\sin 2\alpha - 2\alpha \cos 2\alpha}$$

$$\sigma_{\theta\theta} = 0$$

(d) For α small, simple beam theory would give

$$\sigma = \frac{My}{I} \quad \text{where } y = r\theta$$

$$I = \frac{(2r\alpha)^3}{12} = \frac{2}{3} r^3 \alpha^3$$

$$\therefore \sigma = \frac{3}{2} \frac{M\theta}{r^2 \alpha^3}.$$

$$\sigma_{rr} = \frac{-2M \sin 2\theta}{r^2(\sin 2\alpha - 2\alpha \cos 2\alpha}$$

$$= \frac{-2M \left(2\theta - \cancelto{0}{\frac{8\theta^3}{3!}} + O\,\theta^5\right)}{r^2 \left(\cancel{2\alpha} \underbrace{- \frac{8\alpha^3}{3!} - \cancel{2\alpha} + 2\alpha . \frac{4\alpha^2}{2!}} + O . \alpha^5 \right)}$$

NB. Important to retain 2nd term for both sine + cosine expansions.

$$= \frac{12 M\theta}{8 r^2 \alpha^3}$$

QED