3D7 2007

1) (a) $\underset{\sim}{H}\,\underset{\sim}{r} = p$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} t_{I} \\ t_{II} \\ t_{III} \\ t_{IV} \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} =$$

(b)

2 pivots

Columns without pivots = 'free variabl[es]' correspond to redundant bars

Set $t_{III} = t_{IV} = 0$ and find t_I & t_{II} by back substitution:

$$\frac{1}{\sqrt{2}} t_{II} = 0 \quad \therefore \; t_{II} = 0$$

$$t_I + \frac{1}{\sqrt{2}} t_{II} = W \quad \therefore \; t_I = W$$

so $\underset{\sim}{r}_0 = [W \;\; 0 \;\; 0 \;\; 0]^T$

This is a set of bar forces in equilibrium with the external load.

Set $t_{III} = 1$, $t_{IV} = 0$ and find rest for RHS = 0

$$\frac{1}{\sqrt{2}} t_{II} + 1 = 0 \quad \therefore \; t_{II} = -\sqrt{2}$$

$$t_I + \frac{1}{\sqrt{2}} t_{II} = 0 \quad \therefore \; t_I = 1$$

so $\underset{\sim}{r}_1 = [1 \;\; -\sqrt{2} \;\; 1 \;\; 0]^T$

A state of self stress

1)(b) (cont.) Set $t_{III} = 0$ $t_{IV} = 1$ RHS $= 0$

$$\tfrac{1}{\sqrt{2}} t_{II} + \tfrac{1}{\sqrt{2}} = 0 \quad \therefore \quad t_{II} = -1$$

$$t_I + \tfrac{1}{\sqrt{2}} t_{II} - \tfrac{1}{\sqrt{2}} = 0 \quad \therefore \quad t_I = \tfrac{2}{\sqrt{2}} = \sqrt{2}$$

so $\underset{\sim}{r}_2 = \begin{bmatrix} \sqrt{2} & -1 & 0 & 1 \end{bmatrix}^T$

Another, independent, state of self stress

General solution of equilibrium equations is:

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + \underset{\sim}{S}\,\underset{\sim}{x} \qquad \underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{r}_1 & \underset{\sim}{r}_2 \end{bmatrix}$$

$$= \begin{bmatrix} W \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c) (i) The structure is initially unstressed
$\therefore$ no lack-of-fit and $\underset{\sim}{e}_0 = 0$

$\therefore$ Solve $\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{S}\, \underset{\sim}{x} = -\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{r}_0$

Flexibility matrix

$$\underset{\sim}{F} = \frac{L}{AE} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\underset{\sim}{S}^T = \begin{bmatrix} 1 & -\sqrt{2} & 1 & 0 \\ \sqrt{2} & -1 & 0 & 1 \end{bmatrix}$$

1) (b) (cont.)

$$\underset{\sim}{S}^T \underset{\sim}{F} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \frac{L}{AE}$$

$$\underset{\sim}{S}^T \underset{\sim}{F} \underset{\sim}{S} = \begin{bmatrix} 2+2\sqrt{2} & 2+\sqrt{2} \\ 2+\sqrt{2} & 2+2\sqrt{2} \end{bmatrix} \frac{L}{AE} = \begin{bmatrix} 4.828 & 3.414 \\ 3.414 & 4.828 \end{bmatrix} \frac{L}{AE}$$

$$\underset{\sim}{S}^T \underset{\sim}{F} \underset{\sim}{r}_0 = \begin{bmatrix} W \\ \sqrt{2}W \end{bmatrix} \frac{L}{AE}$$

Hence system of compatibility equations is:

$$\begin{bmatrix} 2+2\sqrt{2} & 2+\sqrt{2} \\ 2+\sqrt{2} & 2+2\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} W$$

$$\therefore \begin{bmatrix} 2+2\sqrt{2} & 2+\sqrt{2} \\ 0 & 2+2\sqrt{2} - (2+\sqrt{2}) \times \frac{(2+\sqrt{2})}{(2+2\sqrt{2})} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ \sqrt{2} - \frac{2+\sqrt{2}}{2+2\sqrt{2}} \end{bmatrix} W$$

$$\begin{bmatrix} 4.828 & 3.414 \\ 0 & 2.414 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0.7071 \end{bmatrix} W$$

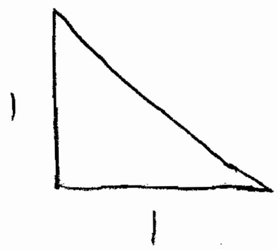
hence $x_2 = -0.293\,W$ and $x_1 = 0$

So

$$\underset{\sim}{r} = \begin{bmatrix} W \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.293W \end{bmatrix} = \begin{bmatrix} 0.586 \\ 0.293 \\ 0 \\ -0.293 \end{bmatrix} W$$

(c) The <u>changes</u> in bar forces can be obtained by setting $\underset{\sim}{e}_0 = [\alpha L \Delta T \;\; 0 \;\; 0 \;\; 0]^T$ and solving $\underset{\sim}{S}^T \underset{\sim}{F} \underset{\sim}{S} \underset{\sim}{x}^* = -\underset{\sim}{S}^T \underset{\sim}{e}_0$ so that $\Delta \underset{\sim}{r} = \underset{\sim}{S} \underset{\sim}{x}$

2)



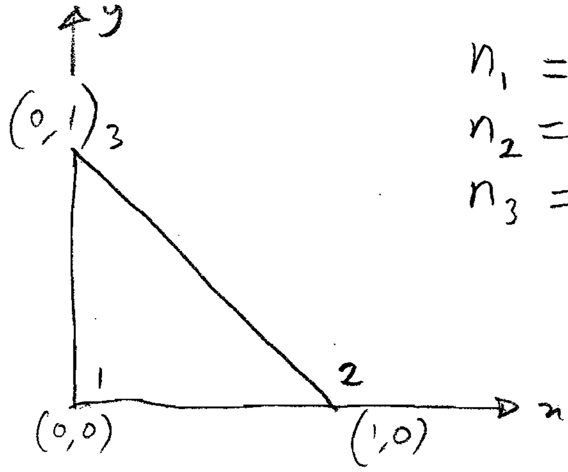
Triangular element mass $= \rho A t$
$= \frac{1}{2}\rho t$

(a) Lumped mass matrix

$$\underset{\sim}{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{6}\rho t$$

For each degree of freedom (in this case 2, x or y) lump $\frac{1}{3}$ of the mass to each node.

2) (cont.)
(b) (i)



$n_1 = (1-x-y)$
$n_2 = x$
$n_3 = y$

$$\therefore N = \begin{bmatrix} n_1 & 0 & n_2 & 0 & n_3 & 0 \\ 0 & n_1 & 0 & n_2 & 0 & n_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-x-y & 0 & x & 0 & y & 0 \\ 0 & 1-x-y & 0 & x & 0 & y \end{bmatrix}$$

(b)(ii)
Consistent mass matrix

$$M = \int_V N^T \rho N \, dV$$

$dV = t\,dx\,dy$
$= t\,dA$

$$= t\rho \int_A \begin{bmatrix} 1-x-y & 0 \\ 0 & 1-x-y \\ x & 0 \\ 0 & x \\ y & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1-x-y & 0 & x & 0 & y & 0 \\ 0 & 1-x-y & 0 & x & 0 & y \end{bmatrix} dA$$

$$= t\rho \int_{dA} \begin{bmatrix} (1-x-y)^2 & 0 & x(1-x-y) & 0 & y(1-x-y) & 0 \\ 0 & (1-x-y)^2 & 0 & x(1-x-y) & 0 & y(1-x-y) \\ x(1-x-y) & 0 & x^2 & 0 & xy & 0 \\ 0 & x(1-x-y) & 0 & x^2 & 0 & xy \\ y(1-x-y) & 0 & xy & 0 & y^2 & 0 \\ 0 & y(1-x-y) & 0 & xy & 0 & y^2 \end{bmatrix}$$

2) (cont.)

(b) (ii)

$$\int_A x^2\, dA = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y} x^2 \, dx\, dy = \int_{y=0}^{y=1} \left[\frac{x^3}{3}\right]_{x=0}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1} \frac{(1-y)^3}{3}\, dy = \frac{1}{3}\int_{y=0}^{y=1} 1 - 3y + 3y^2 - y^3 \, dy$$

$$= \frac{1}{3}\left[y - \frac{3y^2}{2} + y^3 - \frac{y^4}{4}\right]_{y=0}^{y=1}$$

$$= \frac{1}{3}\left\{1 - \frac{3}{2} + 1 - \frac{1}{4}\right\} = \frac{1}{12}$$

and $\int_A y^2\, dA = \int_A (1-x-y)^2\, dA = \frac{1}{12}$

$$\int_A xy\, dA = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y} xy\, dx\, dy = \int_{y=0}^{y=1}\left[\frac{x^2 y}{2}\right]_{x=0}^{x=1-y} dy$$

$$= \int_{y=0}^{y=1} \frac{(1-y)^2 y}{2}\, dy = \frac{1}{2}\int_{y=0}^{y=1} y - 2y^2 + y^3\, dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_{y=0}^{y=1}$$

$$= \frac{1}{2}\left\{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right\} = \frac{1}{24}$$

2) (cont.) (b) (ii)

hence

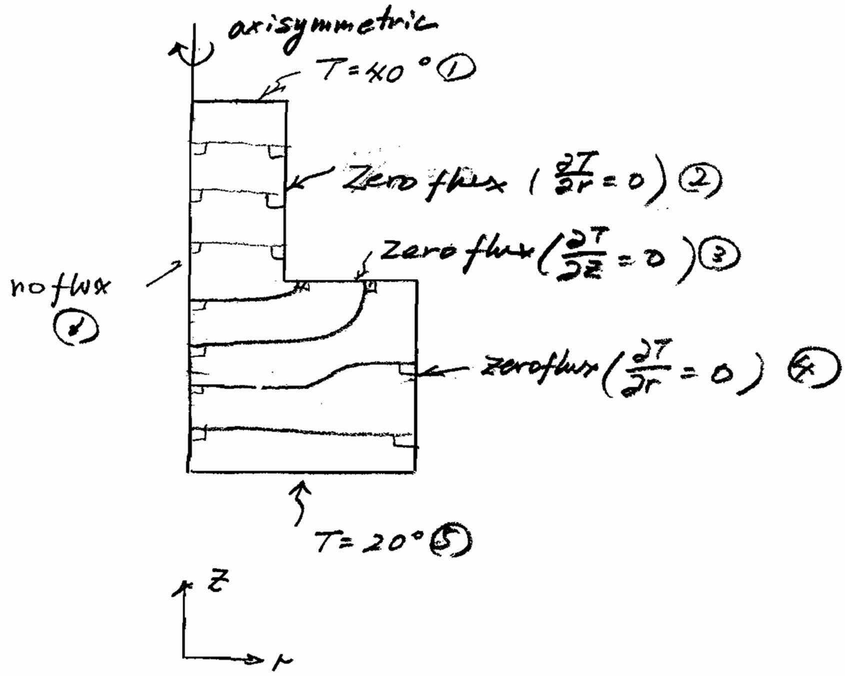
$$M = \frac{tp}{24}\begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

check total mass (per degree of freedom) $= \frac{tp}{24}(3 \times 2 + 6 \times 1)$

$= \frac{1}{2} tp$ as required

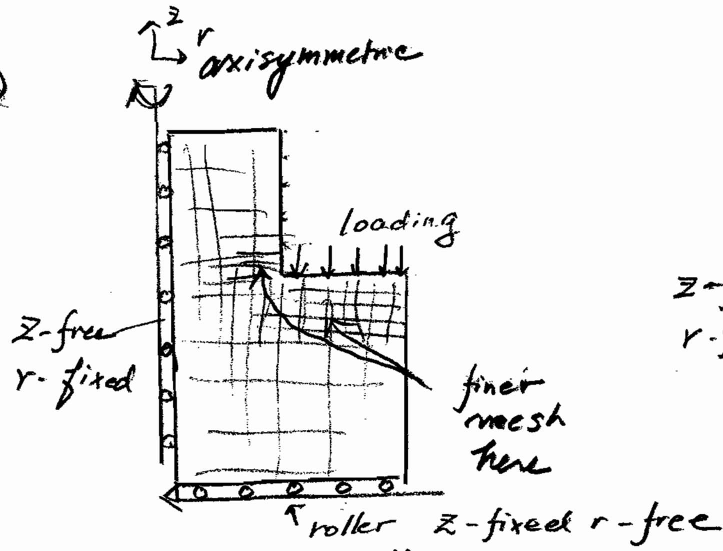
3

(a) (i)

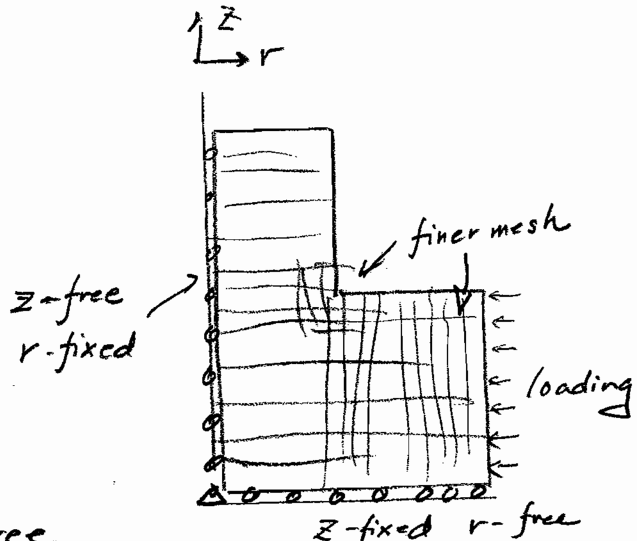


(ii) At ②, ③ & ④ shown above,
The convection boundary condition should be applied instead of zero flux condition. The flux becomes $q = h(T - T_0)$ when h is the heat transfer coefficient and $T_0 = 25\,°C$ in this case. T at the boundary surface will be computed for the finite element analysis.

(b)



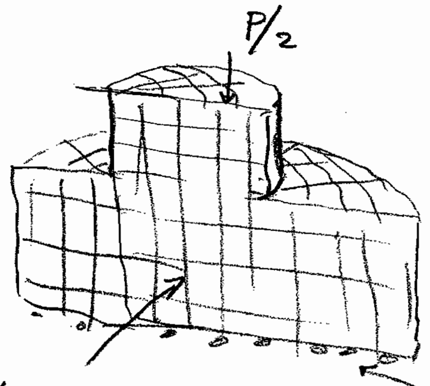
(i) on top of the bottom cylinder



(ii) on the side boundary of the bottom cylinder.

3) (cont.)

(c) 3D model of the half of the object should be used.



fixed zero displacement perpendicular to the surface

rollers at the bottom surface fixed zero displacement perpendicular to the surface.

(d) The Gauss points. The first derivative of the nodal values are most accurate at the Gauss point locations. Stresses are directly related to strains ($\sigma = D\varepsilon$) and the strain are the first derivatives of the nodal displacement values.

4) (a)

Shape functions for element I

$$n_1 = \frac{100 - X}{100} \qquad n_2 = \frac{X - Y}{100} \qquad n_3 = \frac{Y}{100}$$

Strain shape function matrix

$$B_i = \begin{bmatrix} \frac{\partial n_1}{\partial X} & 0 & \frac{\partial n_2}{\partial X} & 0 & \frac{\partial n_3}{\partial X} & 0 \\ 0 & \frac{\partial n_1}{\partial Y} & 0 & \frac{\partial n_2}{\partial Y} & 0 & \frac{\partial n_3}{\partial Y} \\ \frac{\partial n_1}{\partial Y} & \frac{\partial n_1}{\partial X} & \frac{\partial n_2}{\partial Y} & \frac{\partial n_2}{\partial X} & \frac{\partial n_3}{\partial Y} & \frac{\partial n_3}{\partial X} \end{bmatrix}$$

$$= \frac{1}{100} \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

(b)

$$\underset{\sim}{K} = \int B^T D B \, dV$$

$$= \frac{1}{100} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \frac{1}{100} \begin{bmatrix} -1 & 0 & 1 & 0 & \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

4) (cont.)

$$\underset{\sim}{K} = \frac{1}{100}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \frac{1}{100}\begin{array}{cc} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} & \begin{array}{cc} 1 & x \\ & y \\ 2 & x \\ & y \\ 3 & x \\ & y \end{array} \\ \begin{array}{cccccc} 1 & & 2 & & 3 & \\ x & y & x & y & x & y \end{array} & \end{array}$$

(c) Reduced stiffness matrix $\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{P}$ for element

$$\frac{1}{100}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{bmatrix} = \begin{bmatrix} P_{2x} \\ P_{3x} \\ P_{3y} \end{bmatrix}$$

(d)

Reduced K for element II must therefore be:

$$\frac{1}{100}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} d_{4y} \\ d_{3y} \\ u_{3x} \end{bmatrix} = \begin{bmatrix} P_{4y} \\ P_{3y} \\ P_{3x} \end{bmatrix}$$

Rearranging:

$$\frac{1}{100}\begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} d_{4y} \\ d_{3y} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} P_{3x} \\ P_{3y} \\ P_{4y} \end{bmatrix}$$

$$\frac{1}{100}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{bmatrix} d_{3x} \\ d_{3y} \\ d_{4y} \end{bmatrix} = \begin{bmatrix} P_{3x} \\ P_{3y} \\ P_{4y} \end{bmatrix}$$

4) (cont.)

Combining K_I and K_{II} gives the overall stiffness matrix:

$$\frac{1}{100}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2}+1 & 0+0 & 0 \\ 0 & 0+0 & 1+\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \\ d_{4y} \end{bmatrix} = \begin{bmatrix} P_{2x} \\ P_{3x} \\ P_{3y} \\ P_{4y} \end{bmatrix}$$

hence

$$K = \frac{1}{100}\begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \text{ kN/mm}$$