Module 3F1, April 2007 – SIGNALS AND SYSTEMS – Solutions

ENGINEERING TRIPOS PART IIA

Tuesday 1 May 2007 9 to 10.30

Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Module 3F1, April 2007 – SIGNALS AND SYSTEMS – Answers

1 (b) (i) $G(s) = \frac{1}{s(s+1)}$. Both systems are (marginally) unstable.
(ii) $k > 0$.
(c) (i)

$$|z| = \frac{\sqrt{1+2\sigma+\sigma^2+\omega^2}}{\sqrt{1-2\sigma+\sigma^2+\omega^2}}$$

2 (a) (i) $g_k = p^k$, $k = 0, 1, 2, \ldots$.
(ii) $y_k = \sum_{i=0}^{k} u_{k-i} g_i = \sum_{i=0}^{k} u_{k-i} p^i$
(b) (ii) $r_{XX}(t_1, t_2) = \frac{1}{3} r_{UU}(t_1, t_2)$. The random process is WSS.

3 (b) $\Phi_Y(u) = \Phi_{X_1}(u)\Phi_{X_2}(u)$.
(c) $\Phi_{X_1}(u) = \operatorname{sinc}(ub_1/2)$, $\Phi_{X_2}(u) = \operatorname{sinc}^2(ub_2/2)$, $\Phi_Y(u) = \operatorname{sinc}(ub_1/2)\operatorname{sinc}^2(ub_2/2)$.

4 (a) $P(A) = 0.8$, $P(B) = 0.1$, $P(C) = 0.1$.
(b) Efficiency $= 91.92\%$.
(c) $I(X_{n+1}; X_n) = 0.1525$ and $H(X_{n+1}|X_n) = 0.7694$.

1 (a) Rewriting and collecting terms in z we get $(z+1)s = z-1$ or $z(1-s) = 1+s$. Dividing

$$z = \frac{1+s}{1-s}$$

gives the desired result. [10%]

(b) (i) Replacing z in $G(z)$ by $z = (1+s)/(1-s)$ gives

$$G(s) = \frac{\left(\frac{1+s}{1-s}+1\right)^2}{2\frac{1+s}{1-s}\left(\frac{1+s}{1-s}-1\right)} = \frac{1}{s(s+1)}$$

The poles of $G(z)$ are at 0 and 1 (therefore marginally unstable) and the poles of $G(s)$ are at -1 and 0 (also marginally unstable). [25%]

(ii) We can check the closed loop stability of either system. If we look at the continuous-time system, we want the closed-loop poles in the left-half plane, ie. the roots of

$$\frac{1}{s(s+1)}k+1 = 0$$

or $s^2+s+k=0$ must have negative real part. This is true if and only if $k>0$. The same range of k will also stabilise $G(z)$. [25%]

(c) (i) Let $s = \sigma + j\omega$. Then

$$z = \frac{1+\sigma+j\omega}{1-\sigma-j\omega}$$

and

$$\begin{aligned} |z| &= \frac{\sqrt{(1+\sigma)^2+\omega^2}}{\sqrt{(1-\sigma)^2+\omega^2}} \\ &= \frac{\sqrt{1+2\sigma+\sigma^2+\omega^2}}{\sqrt{1-2\sigma+\sigma^2+\omega^2}} \end{aligned}$$

[20%]

(ii) If $|z|<1$ then $1+2\sigma+\sigma^2+\omega^2 < 1-2\sigma+\sigma^2+\omega^2$ or $2\sigma < -2\sigma$ which is equivalent to $\sigma < 0$.

If $\sigma < 0$, then $4\sigma < 0$ or $2\sigma < -2\sigma$. Adding the positive number $1+\sigma^2+\omega^2$ on both sides gives $1+2\sigma+\sigma^2+\omega^2 < 1-2\sigma+\sigma^2+\omega^2$ which means that $|z|<1$. [20%]

2 (a) (i) Using the inverse z-transform $g_k = p^k$, $k = 0, 1, 2, \ldots$. [10%]

(ii) The convolution sum is given by

$$\begin{aligned} y_k &= \sum_{i=0}^{k} u_{k-i} g_i \\ &= \sum_{i=0}^{k} u_{k-i} p^i \end{aligned}$$

[10%]

(iii) We have

$$\begin{aligned} |y_k| &= \left| \sum_{i=0}^{k} u_{k-i} p^i \right| \\ &\leq \sum_{i=0}^{k} |u_{k-i}| \left| p^i \right| \\ &\leq \sum_{i=0}^{k} M \left| p^i \right| \\ &\leq M \sum_{i=0}^{\infty} \left| p^i \right| \\ &\leq M \frac{1}{1 - |p|} \end{aligned}$$

The last inequality is valid when $|p| < 1$. Hence, when $|p| < 1$ the system is stable since a bounded input $\{u_i\}$ produced a bounded output $\{y_i\}$. [30%]

(b) (i) The autocorrelation function of such a process for sampling times, t_1 and t_2, is given by

$$r_{XX}(t_1, t_2) = E[X(t_1, \alpha) X(t_2, \alpha)] = \int \int x_1 x_2 f(x_1, x_2) dx_1 dx_2$$

where the expectation is performed over all α (i.e. the whole ensemble), and $f(x_1, x_2)$ is the joint pdf when x_1 and x_2 are samples taken at times t_1 and t_2 from the same random event α of the random process X.

$X(t, \alpha)$ is WSS if (1) the mean is independent of t, i.e. $E[X(t, \alpha)] = \mu$ for all t; and (2) the autocorrelation depends only upon $\tau = t_2 - t_1$, i.e. $E[X(t_1, \alpha) X(t_2, \alpha)] = E[X(t_1, \alpha) X(t_1 + \tau, \alpha)] = r_{XX}(\tau)$ for all t_1. [20%]

(TURN OVER for continuation of Question 2

(ii) The autocorrelation function of $X(t,\alpha)$ is given by

$$\begin{aligned} r_{XX}(t_1,t_2) &= E[A(\alpha)U(t_1)A(\alpha)U(t_2)] \\ &= E[A^2(\alpha)]E[U(t_1)U(t_2)] \\ &= \frac{1}{3}r_{UU}(t_1,t_2) \\ &= \frac{1}{3}r_{UU}(\tau) \end{aligned}$$

In addition, $E[A(\alpha)U(t)] = E[A(\alpha)]U(t) = 0$. Hence, $X(t,\alpha)$ is WSS. [30%]

3 (a) $Y = X_1 + X_2$. We can write the joint pdf for y and x_1 by rewriting the conditional probability formula:

$$f(y, x_1) = f(y|x_1) f_1(x_1)$$

It is clear that the event 'Y takes the value y conditional upon $X_1 = x_1$' is equivalent to X_2 taking a value $y - x_1$ (since $X_2 = Y - X_1$). Hence,

$$f(y|x_1) = f_2(y - x_1)$$

Now, $f(y)$ may be obtained using the marginal probability formula. Hence,

$$\begin{aligned} f(y) &= \int f(y|x_1) f_1(x_1) dx_1 \\ &= \int f_2(y - x_1) f_1(x_1) dx_1 \\ &= f_2 * f_1 \quad \text{(Convolution)} \end{aligned}$$

[30%]

(b) The characteristic function of a pdf is defined as

$$\Phi_Y(u) = E[e^{juY}] = \int_{-\infty}^{\infty} e^{juy} f_Y(y) dy = \mathscr{F}(-u)$$

where $\mathscr{F}(u)$ is the Fourier Transform of the pdf. Thus, properties of Fourier Transforms apply. In the case of $Y = X_1 + X_2$, f_Y is given by the convolution of f_2 with f_1, i.e. $f_Y = f_2 * f_1$ which means that $\Phi_Y(u) = \Phi_{X_1}(u)\Phi_{X_2}(u)$ (multiplication of the characteristic functions of X_1 and X_2). [30%]

(c) Using the Data Book

$$\Phi_{X_1}(u) = \frac{1}{b_1} b_1 \operatorname{sinc}\left(\frac{(-u)b_1}{2}\right) = \operatorname{sinc}\left(\frac{ub_1}{2}\right)$$

and

$$\Phi_{X_2}(u) = \frac{1}{b_2} b_2 \operatorname{sinc}^2\left(\frac{(-u)b_2}{2}\right) = \operatorname{sinc}^2\left(\frac{ub_2}{2}\right)$$

Thus,

$$\begin{aligned} \Phi_Y(u) &= \Phi_{X_1}(u)\Phi_{X_2}(u) \\ &= \operatorname{sinc}\left(\frac{ub_1}{2}\right) \cdot \operatorname{sinc}^2\left(\frac{ub_2}{2}\right) \end{aligned}$$

[40%]

(TURN OVER

4 (a) For a valid joint probability table, all the entries should sum to unity. In this case

$$\begin{aligned} \text{sum of all entries} &= 0.72+4\times 0.04+4\times 0.03 \\ &= 0.72+0.16+0.12 \\ &= 1.0 \end{aligned}$$

Hence, the table is valid.

The mean probability of each state is given by the sum across each row of the table. Hence

$$\begin{aligned} \text{Probability of state A} &= 0.72+0.04+0.04=0.8 \\ \text{Probability of state B} &= 0.04+0.03+0.03=0.1 \\ \text{Probability of state C} &= 0.04+0.03+0.03=0.1 \end{aligned}$$

[15%]

(b) A Huffman code is:

```
Code Symbol Probability

1      AA  0.72------------------------------------\
                                                    \
0111   AB  0.04--\1                                  \1
                  --0.08--\                           \
0110   AC  0.04--/0        \1                          --1.0
                            --0.16--\                 /
0101   BA  0.04--\1        /0        \               /
                  --0.08--/           \1            /0
0100   CA  0.04--/0                    \           /
                                        --0.28--/
0011   BB  0.03--\1                    /
                  --0.06--\           /0
0010   BC  0.03--/0        \1        /
                            --0.12--/
0001   CB  0.03--\1        /0
                  --0.06--/
0000   CC  0.03--/0
```

(cont.

Mean length for 2 symbols $= 0.72 + 4 \times 0.04 \times 4 + 4 \times 0.03 \times 4 = 0.72 + 0.64 + 0.48$. Entropy $= -[0.72\log_2(0.72) + 0.16\log_2(0.04) + 0.12\log_2(0.03)] = 0.3412 + 0.7430 + 0.6071 = 1.6913$. Hence

$$\text{Efficiency} = \frac{1.6913}{1.84} = 91.92\%$$

Efficiency will be worse for a Huffman code for single symbols because the Probability of the most probable event A increases to 0.8 and we do not take account of the strong correlations between adjacent symbols. [35%]

(c) $I(X_{n+1};X_n) = H(X_{n+1}) - H(X_{n+1}|X_n)$

$H(X_{n+1}) = -[0.8\log_2(0.8) + 2 \cdot 0.1\log_2(0.1)] = 0.2575 + 0.6644 = 0.9219$

$H(X_{n+1}|X_n) = 0.8H(X_{n+1}|X_n = A) + 0.1H(X_{n+1}|X_n = B) + 0.1H(X_{n+1}|X_n = C)$

Now $H(X_{n+1}|X_n = A) = H\left(\frac{0.72}{0.8}, \frac{0.04}{0.8}, \frac{0.04}{0.8}\right) = H(0.9, 0.05, 0.05) = -[0.9\log_2(0.9) + 2 \cdot 0.05\log_2(0.05)] = 0.1368 + 0.4322 = 0.5690$

$H(X_{n+1}|X_n = B) = H(X_{n+1}|X_n = C) = H\left(\frac{0.04}{0.1}, \frac{0.03}{0.1}, \frac{0.03}{0.1}\right) = H(0.4, 0.3, 0.3) = -[0.4\log_2(0.4) + 2 \cdot 0.3\log_2(0.3)] = 0.5288 + 1.0422 = 1.5710$

Then, the conditional entropy is $H(X_{n+1}|X_n) = 0.8 \cdot 0.5690 + 0.2 \cdot 1.5710 = 0.4552 + 0.3142 = 0.7694$

and the mutual information is $I(X_{n+1};X_n) = 0.9219 - 0.7694 = 0.1525$ bits. [35%]

(d) Arithmetic Coding is the best way to achieve a bit rate very close to any number of symbols. The algorithm maintains two values which are the lower and upper ends of a range starting at 0 and 1. This range is then partitioned into sub-ranges according to the probability of each symbol. The appropriate sub-range is then selected to become the new range according to the symbol emitted by the source.

When the message is complete, the fraction is converted to binary and the appropriate number of bits are transmitted over the channel. The precision required (and hence the number of bits) is determined by the size of the final sub-range, which is just the product of the probabilities of all the symbols in the message. [15%]

END OF PAPER