PART IIA 2007
3F2 Systems and control
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3F2 Systems and Control: 2007 Solutions

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1. (a) $x(t) = e^{At}x(0)$ is the solution to the equation $\dot{x} = Ax$ when there is no input, where the matrix exponential function is defined as

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \tag{1}$$

as can be checked by substituting back in the differential equation.
When there is an input, the effect on the state trajectory is given by the convolution integral (assuming $x(0) = 0$):

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

(If this formula is not remembered it can be obtained by the 'integrating factor' method: multiply both sides of $\dot{x} - Ax = Bu$ by e^{-At} to get $\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu(t)$ then integrate both sides.)
When there is an input signal and a non-zero initial condition, their effects can be added together, since we have a linear system:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \tag{3}$$

(b) $W^{-1}AW = \Lambda \Rightarrow A = W\Lambda W^{-1}$. Therefore

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots && (4)\\ &= WW^{-1} + W\Lambda W^{-1}t + \frac{(W\Lambda W^{-1}t)^2}{2!} + \frac{(W\Lambda W^{-1}t)^3}{3!} + \cdots && (5)\\ &= W[I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \cdots]W^{-1} && (6)\\ &= We^{\Lambda t}W^{-1} && (7) \end{aligned}$$

where in the third line we used the fact that $(WXW^{-1})^k = WX^kW^{-1}$.
This can be used to calculate e^{At} because, since Λ is diagonal,

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{8}$$

(as can be checked from the definition of the matrix exponential).

(c) From the previous part we have

$$e^{At} = We^{\Lambda t}W^{-1}$$
$$= \begin{bmatrix} -1 & 0.001j & -0.001j \\ -0.05 & 1 & 1 \\ -0.0002 & 0.02j & -0.02j \end{bmatrix} \begin{bmatrix} e^{-0.1t} & 0 & 0 \\ 0 & e^{(-2+8.9j)t} & 0 \\ 0 & 0 & e^{(-2-8.9j)t} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0.05 \\ -0.025 & 0.5 & -25j \\ -0.025 & 0.5 & 25j \end{bmatrix} \tag{9}$$

To get the (3,1) element we need consider only the third row of the first matrix and the first column of the last matrix:

$$\begin{aligned}
(e^{At})_{3,1} &= \begin{bmatrix} -0.0002 & 0.02j & -0.02j \end{bmatrix} \begin{bmatrix} e^{-0.1t} & 0 & 0 \\ 0 & e^{(-2+8.9j)t} & 0 \\ 0 & 0 & e^{(-2-8.9j)t} \end{bmatrix} \begin{bmatrix} -1 \\ -0.025 \\ -0.025 \end{bmatrix} && (10) \\
&= \begin{bmatrix} -0.0002 & 0.02j & -0.02j \end{bmatrix} \begin{bmatrix} e^{-0.1t} \\ -0.025e^{(-2+8.9j)t} \\ -0.025e^{(-2-8.9j)t} \end{bmatrix} && (11) \\
&= -0.0002e^{-0.1t} - 0.0005je^{(-2+8.9j)t} + 0.0005je^{(-2-8.9j)t} && (12) \\
&= -0.0002e^{-0.1t} - 0.0005je^{-2t}(e^{8.9jt} - e^{-8.9jt}) && (13) \\
&= -0.0002e^{-0.1t} + 0.001e^{-2t}\sin(8.9t) && (14)
\end{aligned}$$

A surprising number of candidates omitted the "t" from their solutions, so they gave answers which were not functions of time.

The physical meaning of the (3,1) element is that it gives the response of the pitch rate (x_3) to a perturbation in the forward velocity (x_1).

Many candidates stated here that e^{At} relates $\dot{x}$ to x, rather than $x(t)$ to $x(0)$ — despite answering part (a) correctly. Also many confused $(3,1)$ with $(1,3)$.

(d) All three eigenvalues have negative real parts, so the state vector will settle to an equilibrium value after the elevator has settled to a constant angle. Since two of the eigenvalues are complex, the transient behaviour will contain a decaying oscillation with frequency 8.9 rad/sec (a period of approx 0.7 sec), and a damping factor $c = 2/\sqrt{2^2 + 8.9^2} = 0.22$. This oscillation will decay with a time constant of 0.5 sec, whereas the real eigenvalue will result in a slower exponential decay with a time constant of 10 sec.

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2. (a) From the 'hence' part of the question, we should aim to show that the root-locus lies entirely on the imaginary axis.

There are 2 zeros and 4 poles, so there are 2 asymptotes, making angles of $\pm\pi/2$ rad with the real axis. Since all the zeros and poles are on the imaginary axis, the asymptotes lie on the imaginary axis too ('centre of gravity' rule).

Consider a point on the imaginary axis between 0 and ja. The 'angle contributions' from the poles and zeros at this point are:
From the poles at 0: $+\pi/2$ rad from each.
From the pole at jb: $-\pi/2$.
From the zero at ja: $+\pi/2$. (Subtract contributions from zeros.)
From the pole at $-jb$: $+\pi/2$.
From the zero at $-ja$: $-\pi/2$.
Sum of angle contributions: π rad. Hence the point under consideration is on the root-locus (angle condition).

Consider a point on the imaginary axis between ja and jb. The angle contributions are as above, except:
From the zero at ja: $-\pi/2$.
Sum of angle contributions: 0 rad. Hence the point is not on the root-locus.

Consider a point on the imaginary axis above jb. The angle contributions are as above, except:
From the pole at jb: $+\pi/2$.
From the zero at ja: $-\pi/2$.
Sum of angle contributions: π rad. Hence the point is on the root-locus.

Hence the root-locus diagram has the form shown in Fig. 1. It lies entirely on the imaginary axis. So all the closed-loop poles are purely imaginary for all positive values of k.

Very few candidates attempted to check the angle condition, which is very easy here.

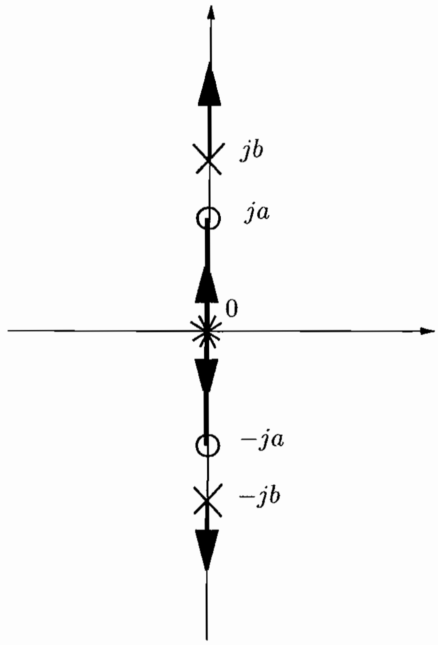

Figure 1: Root-locus diagram for Question 2(a)

(b) Now we need to sketch the root-locus diagram for the return-ratio

$$\frac{s+z}{s+p} \times \frac{s^2+a^2}{Js^2(s^2+b^2)}. \tag{15}$$

First note that there are still 2 asymptotes, but these are now to the left of the imaginary axis, since the 'centre of gravity' of the poles and zeros is to the left of 0 (since $z < p$; recall that the asymptotes meet at $(\sum(poles) - \sum(zeros))/(\#poles - \#zeros)$).

Consider any point on the root-locus diagram in part 2a. The angle contributions at this point are as above, but in addition there is a contribution $+\psi$ from the pole at $-p$, and a contribution $-\theta$ from the zero at $-z$ — see Fig. 2. So the total angle contribution is $\pi + \psi - \theta < \pi$, since $z < p \Rightarrow \psi < \theta$. Thus the point considered is not on the root-locus now. We need to show that we will find a point on the root-locus if we move to the left of the imaginary axis, but not if we move to the right.

We need a net positive change in the angle contributions from the plant poles and zeros, to compensate for the negative contribution from the controller pole-zero pair. Consider a point between 0 and ja. From Figure 3 we see that if we move to the left the net change in angle contributions from the plant poles and zeros is $2\alpha - (\beta - \gamma) + (\delta - \eta)$. But $\delta - \eta > \beta - \gamma$, so the net change is positive, as required. If we move to the right the net change is negative, by a similar argument. If the point considered is above jb (see Fig. 4) then the net change is $2\alpha - (\beta - \gamma) + (\eta - \delta)$, and $\eta - \delta > \beta - \gamma$, so the same conclusion holds. There is one more detail to check: consider a point on the imaginary axis between ja and jb. Without the lead compensator, the net angle contribution is 0 (as shown above). The net change due to the lead compensator must be smaller than π in magnitude; it is therefore impossible for the root-locus to cross the imaginary axis at such a point.

Hence the complete root-locus plot has the form shown in Fig.5.

Thus we have shown that the root-locus is located entirely in the left half-plane, and hence the closed loop is asymptotically stable for all positive values of k.

A vaguer argument to the effect that 'zeros attract the root-locus' was used by most candidates, backed up by the fact that the asymptotes now lie to the left of the imaginary axis. Arguments along these lines were accepted, though not with full credit.

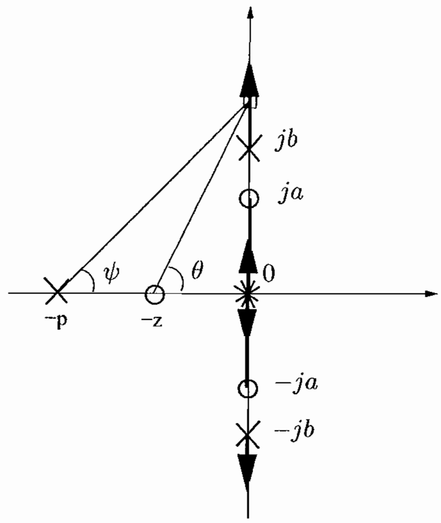

Figure 2: Additional angle contributions for point previously on the root locus.

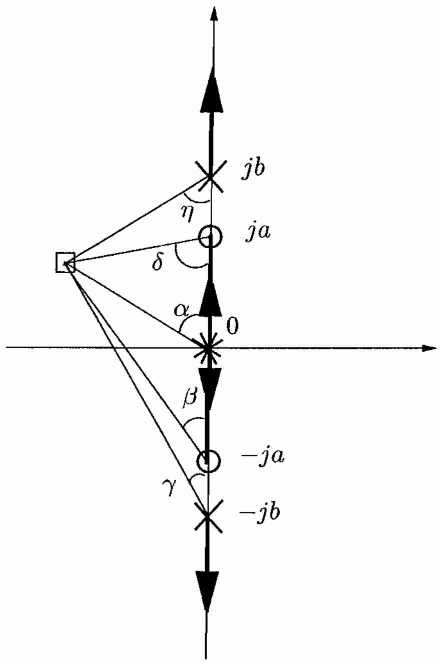

Figure 3: Changes of angle contributions for point located between 0 and ja.

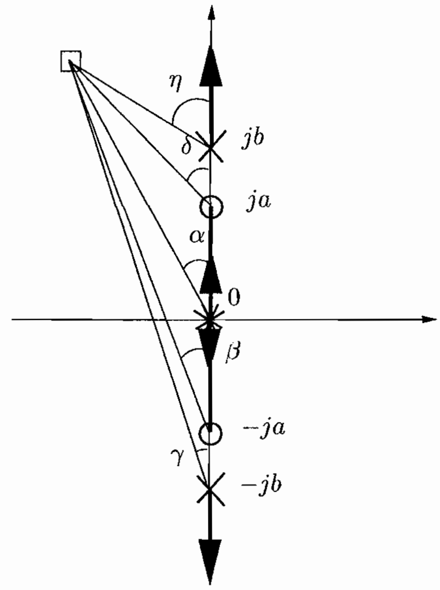

Figure 4: Changes of angle contributions for point located above jb.

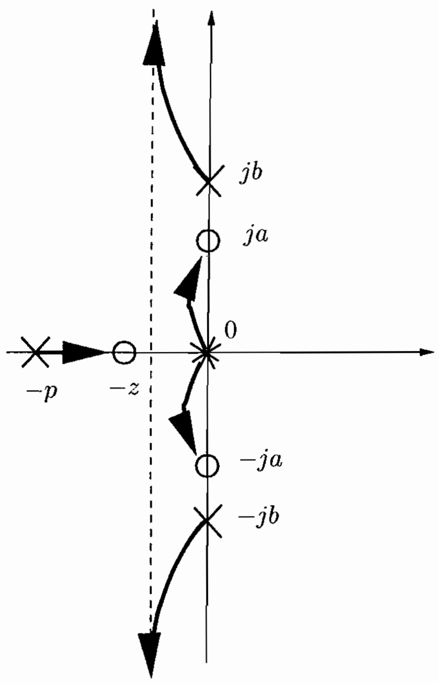

Figure 5: Root-locus diagram for question 2(b).

(c) Now we need to sketch the root-locus diagram for the return-ratio

$$\frac{s+z}{s+p} \times \frac{s^2+a^2}{Js^2(s^2+b^2)} \times \frac{s^2+c^2}{s^2+d^2}. \tag{16}$$

First consider the return-ratio with the lead compensator omitted:

$$\frac{s^2+a^2}{Js^2(s^2+b^2)} \times \frac{s^2+c^2}{s^2+d^2} \tag{17}$$

Since $c < b$ and $d > a$, the pole-zero pattern of this is similar to that in part 2a, in that each pair of poles on the imaginary axis is separated by a zero — see Fig. 6. Thus by the same argument as in part 2a, the root-locus for this return-ratio lies entirely on the imaginary axis, as shown in Fig. 6.
Now adding the lead compensator has the effect of 'pulling' each part of the root-locus diagram to the left, by the same argument as was used in part 2b. The root-locus diagram has the form shown in Fig.7. Thus the closed-loop system is asymptotically stable for each positive value of k.

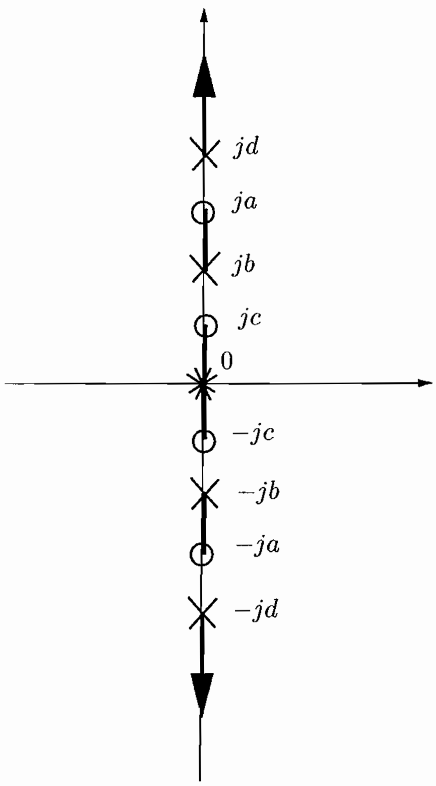

Figure 6: Root-locus diagram for part 2c without the lead compensator.

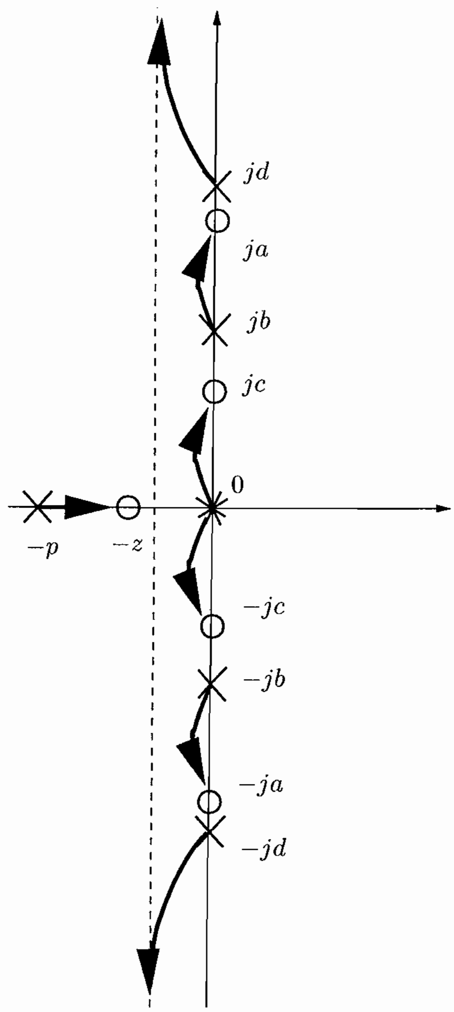

Figure 7: Root-locus diagram for Question 2(c).

3. (a) Applying Newton's second law to each of the masses gives:

$$m_1\ddot{y}_1 = u + c(\dot{y}_2 - \dot{y}_1) + k(y_2 - y_1) \quad (18)$$
$$m_2\ddot{y}_2 = -u - c(\dot{y}_2 - \dot{y}_1) - k(y_2 - y_1) \quad (19)$$
$$(20)$$

Letting $x = [y_1, \dot{y}_1, y_2, \dot{y}_2]^T$ gives

$$\dot{x} = \begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -c/m_1 & k/m_1 & c/m_1 \\ 0 & 0 & 0 & 1 \\ k/m_2 & c/m_2 & -k/m_2 & -c/m_2 \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix} u \quad (21)$$

Very few candidates remembered their IA and IB courses here, so most wrote down the wrong equations. Strangely, most knew that the spring force depended on $y_1 - y_2$, but thought that the damper force depended on $\dot{y}_1$ or $\dot{y}_2$ only. Luckily for the candidates, the most common errors did not change the controllability properties in part (c).

(b) For a linear system $\dot{x} = Ax + Bu$, with state dimension n, the controllability matrix is defined as

$$\Gamma = [B, AB, A^2B, \ldots, A^{n-1}B] \quad (22)$$

Controllability test: The system is controllable if and only if $\text{rank}(\Gamma) = n$.

(c) Now we have $c = k = m_1 = m_2 = 1$. Hence we have:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad (23)$$

and hence

$$AB = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad A^2B = A(AB) = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix} \quad (24)$$

and

$$A^3B = A(A^2B) = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \quad (25)$$

Hence we have (since $n = 4$):

$$\Gamma = \begin{bmatrix} 0 & 1 & -2 & 2 \\ 1 & -2 & 2 & 0 \\ 0 & -1 & 2 & -2 \\ -1 & 2 & -2 & 0 \end{bmatrix} \quad (26)$$

from which it is seen by inspection that

row 3 = − row 1 and row 4 = − row 2.

Hence Γ has at most 2 independent rows, so its rank is at most 2. Thus the system is *not controllable* (since $n = 4$).

Note: This could also be done by evaluating the determinant of Γ, but that would take quite a long time. Expansion by the first (or last) column would require only 2 3×3 determinants to be evaluated.

(d) The combined centre of mass of the two masses is at

$$\frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{1}{m_1 + m_2}[m_1, 0, m_2, 0]x. \quad (27)$$

Since the combined system of the two masses has no external force acting on it, its centre of mass remains unchanged, whatever internal force u is applied. Hence

$$[m_1, 0, m_2, 0]x = 0 \quad \text{for all } u. \tag{28}$$

Thus the input u can only move the state vector in directions orthogonal to $[m_1, 0, m_2, 0]^T$.
But the input is known to move the state vector in directions which are in the column span of Γ. Thus all the columns should be orthogonal to $[m_1, 0, m_2, 0]^T$.

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4. (a) Since $r = [0, 1]x$, we have $C = [0, 1]$. Apply the observability test (with $n = 2$):

$$\begin{aligned}
\text{rank}\begin{bmatrix} C \\ CA \end{bmatrix} &= \text{rank}\begin{bmatrix} 0 & 1 \\ c\sigma & a_{22} \end{bmatrix} && (29)\\
&= 2 \quad \text{if and only if} \quad \sigma \neq 0. && (30)
\end{aligned}$$

(b) In general the observer equation is

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) && (31)\\
&= (A - LC)\hat{x} + Bu + Ly && (32)
\end{aligned}$$

The observer poles are the eigenvalues of $A - LC$. In the given system we have $u = \delta$, $y = r$ and $L = [\ell_1, \ell_2]^T$, so that

$$\begin{aligned}
A - LC &= \begin{bmatrix} a_{11} & -1 + c\sigma \\ c\sigma & a_{22} \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} && (33)\\
&= \begin{bmatrix} a_{11} & -1 + c\sigma \\ c\sigma & a_{22} \end{bmatrix} - \begin{bmatrix} 0 & \ell_1 \\ 0 & \ell_2 \end{bmatrix} && (34)\\
&= \begin{bmatrix} a_{11} & -1 + c\sigma - \ell_1 \\ c\sigma & a_{22} - \ell_2 \end{bmatrix} && (35)
\end{aligned}$$

We want the eigenvalues of $A - LC$ to be at a_{11} and a_{22}.
Either:

$$\begin{aligned}
\det[sI - (A - LC)] &\equiv (s - a_{11})(s - a_{22}) && (36)\\
\Rightarrow (s - a_{11})(s - a_{22} + \ell_2) - c\sigma(-1 + c\sigma - \ell_1) &\equiv s^2 - (a_{11} + a_{22})s + a_{11}a_{22} && (37)
\end{aligned}$$

which implies $\boxed{\ell_1 = -1 + c\sigma,\ \ell_2 = 0}$.
Or: Use 'trace = sum of eigenvalues' and 'determinant = product of eigenvalues':

$$\begin{aligned}
a_{11} + a_{22} - \ell_2 = a_{11} + a_{22} \quad &\Rightarrow \quad \ell_2 = 0 && (38)\\
a_{11}(a_{22} - \ell_2) - c\sigma(-1 + c\sigma - \ell_1) = a_{11}a_{22} \quad &\Rightarrow \quad \ell_1 = -1 + c\sigma. && (39)
\end{aligned}$$

(c) Block-diagram of observer/state-feedback combination: standard.
How to ensure closed-loop stability: Design a stable observer and a stable state-feedback scheme, ie ensure that eigenvalues of $A - LC$ and eigenvalues of $A - BK$ all have negative real parts — since the closed-loop eigenvalues are the union of the observer and the state-feedback eigenvalues.

(d) The major limitations on the closed-loop pole positions are:

- Actuator power: Very fast eigenvalues of $A - BK$ would require large gains in K, which would result in saturation of the actuators, causing very nonlinear operation.
- Sensor noise: Very fast eigenvalues of $A - LC$ would require large gains in L, causing sensor noise (in this case rate gyro noise) to be amplified and passed through to the actuators, thus wasting actuator energy, causing wear and tear, and further increasing the likelihood of saturation.

Module 3F2: Systems and Control
Answers 2007

1. (a) —
   (b) $e^{At} = We^{\Lambda t}W^{-1}$
   (c) $-0.0002e^{-0.1t} + 0.001e^{-2t}\sin(8.9t)$
   (d) —
2. —
3. (a)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -c/m_1 & k/m_1 & c/m_1 \\ 0 & 0 & 0 & 1 \\ k/m_2 & c/m_2 & -k/m_2 & -c/m_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix} u$$

   (b) —
   (c) —
   (d) —
4. (a) —
   (b) $L = [-1 + c\sigma, \quad 0]^T$.
   (c) —
   (d) —