PART IIA 2007
3F4 Data transmission
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Module 3F4 '07 Solutions

1.) (a) To reduce the effect of intersymbol interference (ISI) and so improve bit error rate (BER) performance.

(i) The zero forcing equaliser aims to completely eliminate ISI at the sampling instants. To achieve this the equaliser response is,

$$H_E(z) = \frac{1}{P(z)}$$

where $P(z)$ is the z-transform of the sampled received pulse response $p(n)$.

(ii) The ZF equaliser can give rise to unwanted noise amplification if the channel frequency response has nulls at particular frequencies. In this case the gain of the equaliser becomes high at these frequencies amplifying the noise present at its input. To overcome this problem the MMSE equaliser aims to minimise the error between the received symbols and the transmitted symbols thus explicitly including the effect of noise in the design process. Thus a compromise between ISI reduction and noise enhancement is achieved

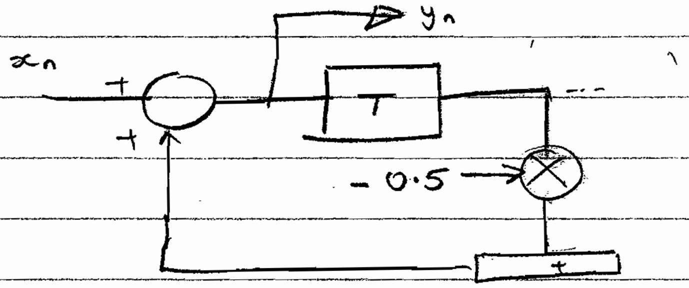
(iii) In a DFE, delayed weighted detected symbols are fed-back and used to cancel ISI. Since detected symbols are used they are noise free and so noise enhancement does not occur. Problems do occur if an error is made about a received symbol. In this case burst errors occur at the equaliser output.

1.) (b) Required equaliser response is,

$$H_E(z) = \frac{1}{P(z)}$$

$$= \frac{1}{1 + 0.5z^{-1}}$$

which is an infinite impulse response filter of form,



ie, $y_n = x_n + 0.5\, y_{n-1}$

$$Y(z) = X(z) - 0.5z^{-1}Y(z)$$

$$Y(z) + 0.5z^{-1}Y(z) = X(z)$$

$$Y(z)(1 + 0.5z^{-1}) = X(z)$$

$$H_E(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 + 0.5z^{-1})}$$

Problems
- stability not guaranteed,
- recursive nature makes them prone to numerical instability
- adaptive methods hard to derive.

(c) (i) For ZF, $H_E(z) = \frac{1}{P(z)}$

$$H_E(z) = \frac{1}{1 + 0.5z^{-1}}$$

From polynomial division

- Alternatively could expand $(1 + 0.5z^{-1})^{-1}$ using Binomial Theorem.

$$
\begin{array}{r|l}
 & 1 - 0.5z^{-1} + 0.25z^{-2} - 0.125z^{-3} \\
\hline
1 + 0.5z^{-1} & 1 \\
 & 1 + 0.5z^{-1} \\
\hline
 & -0.5z^{-1} \\
 & -0.5z^{-1} - 0.25z^{-2} \\
\hline
 & +0.25z^{-2} \\
 & 0.25z^{-2} + 0.125z^{-3} \\
\hline
 & -0.125z^{-3} \\
 & -0.125z^{-3} - 0.0625z^{-4} \\
\hline
 & 0.0625z^{-4}
\end{array}
$$

(2)

Coefficient values are

[1, -0.5, 0.25, -0.125]

(ii) The equalized pulse response is the convolution of the pulse response [1, 0.5] with the FIR equaliser response [1, -0.5, 0.25, -0.125]

1, -0.5, 0.25, -0.125

[0.5, 1] →

1 0 0 0 -0.0625

For unipolar scheme,

worst case '1' = 1 - 0.0625

= 0.9375

worst case '0' = 0

∴ worst case eye opening, h = 0.9375 - 0

h = 0.9375V.

rms noise voltage at equalizer output,

$\sigma_W = \sigma\sqrt{1^2 + 0.5^2 + 0.25^2 + 0.125^2}$

$\sigma_W = 1.15244\,\sigma$

$\sigma_W = 1.15244 \times 0.2$

$\sigma_W = 0.2305\ V$

worst case BER $= Q\left(\frac{h}{2\sigma_W}\right)$

$= Q\left(\frac{0.9375}{2\times 0.2305}\right)$

$= Q(2.03) \approx 0.0228$

From formula,

BER ≈ 0.0212

$\left[\frac{0.127}{6}\right]$

(3)

Without equaliser,

worst case '1' = 1

worst case '0' = 0 + 0.5

∴ worst case eye opening $h = 1 - 0.5 = 0.5V$.

$$BER = Q\left(\frac{h}{2\sigma}\right)$$

$$= Q\left(\frac{0.5}{2 \times 0.2}\right)$$

$$= Q(1.25)$$

From formula,

$BER \approx 0.1058$

$\boxed{\frac{0.458}{4.32}}$

Equaliser gives an improvement of 5 in the BER.

4

2.) (a) (i) Nearest neighbour decoding
The decoder contains a table of all the possible valid received codewords. The received codeword is compared with all the possible valid codewords and the estimated received codeword is the one with the minimum Hamming distance to the received codeword.

(ii) Clearly the process in (a) becomes difficult and time consuming for a large code. Syndrome decoding reduces decoding complexity by associating a unique syndrome to each error pattern (within the error correcting ability of the code). This syndrome is independent of the actual transmitted codeword and is calculated by multiplying a received codeword with the so called parity check matrix.
Having found the syndrome and identified the corresponding error pattern, this pattern is added (modulo-2) to the received codeword in order to correct the bit(s) in error.

(5)

2.) (b)(i) For a code to be linear the mod-2 sum of any 2 codewords should also be a codeword. i.e.,

$$c_3 = c_1 + c_2$$

$$(c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}) = (c_{11}, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}) + (c_{21}, c_{22}, c_{23}, c_{24}, c_{25}, c_{26})$$

so, $c_{31} = c_{11} + c_{21}$
$c_{32} = c_{12} + c_{22}$
$c_{33} = c_{13} + c_{23}$

now,

$$\begin{aligned} c_{34} &= c_{14} + c_{24} \\ &= (d_{11} + d_{13}) + (d_{21} + d_{23}) \\ &= d_{11} + d_{21} + d_{13} + d_{23} \\ &= c_{11} + c_{21} + c_{13} + c_{23} \\ &= c_{31} + c_{33} \end{aligned}$$

$$\begin{aligned} c_{35} &= c_{15} + c_{25} \\ &= (d_{11} + d_{12}) + (d_{21} + d_{22}) \\ &= d_{11} + d_{21} + d_{12} + d_{22} \\ &= c_{11} + c_{21} + c_{12} + c_{22} \\ &= c_{31} + c_{32} \end{aligned}$$

$$\begin{aligned} c_{36} &= c_{16} + c_{26} \\ &= (d_{12} + d_{13}) + (d_{22} + d_{23}) \\ &= d_{12} + d_{22} + d_{13} + d_{23} \\ &= c_{12} + c_{22} + c_{13} + c_{23} \\ &= c_{32} + c_{33} \end{aligned}$$

consequently $(c_{31}, c_{32}, c_{33}, c_{34}, c_{35}, c_{36}) \in C$ and therefore C is a linear code

[Alternatively could find G and show $c = dG$
$\therefore c_2 + c_1 = d_2 G + d_1 G = (d_2 + d_1) G = c_3$
which is a valid code when $d_3 = d_2 + d_1$]

⑥

(ii) Systematic code has generator matrix of form

$G = IP$ ∴

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Now,

$$H = [-P^T \vdots I]$$

∴

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

⑦

(c) The expected number of errors in a 6-bit codeword

$= P(0) \times 0 + P(1) \times 0 + P(2) \times 3 + P(\geq 3) \times 6$

A single error correcting code so zero decoded errors for 0 or 1 received bit in error

2 received bits in error yield 3 decoded bit errors

for more than 2 received bits in error make worst case assumption that all 6 decoded bits are in error.

$P(0)$ = prob of 0 errors received $= (1-0.02)^6 = 0.88584$

$P(1)$ = " " 1 error received $= 6 \times 0.02 \times (1-0.02)^5 = 0.10847$

$P(2)$ = " " 2 " " $= \dfrac{5 \times 6}{2} \times 0.02^2 \times (1-0.02)^4$

$= 5.5342 \times 10^{-3}$

$P(\geq 3)$ = prob of 3 or more errors

$= 1 - P(0) - P(1) - P(2) = 1.558 \times 10^{-4}$

Expected no. of errors $= 0 + 0 + 3 \times 5.5342 \times 10^{-3} + 6 \times 1.558 \times 10^{-4}$

$= 0.0175374$

$\therefore$ BER $= \dfrac{\text{expected no. of errors}}{\text{no. of bits sent}}$

$= \dfrac{0.0175374}{6}$

$= 2.923 \times 10^{-3}$

ie, 6.84 times better than raw BER.

3 (a)

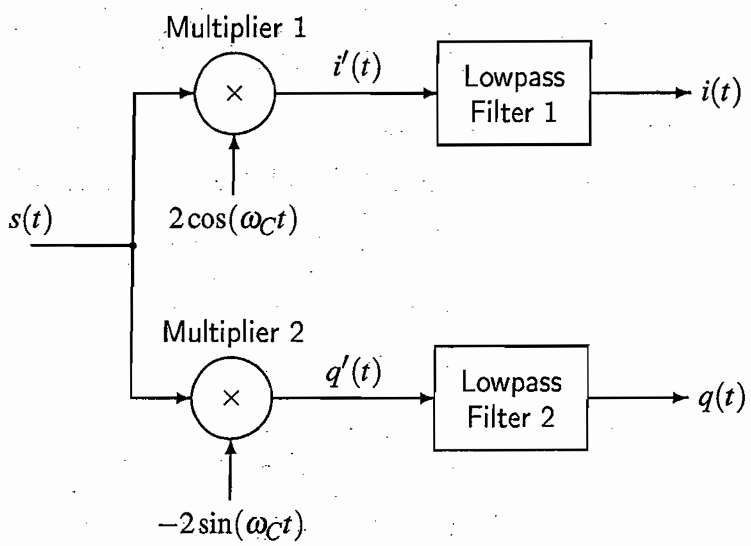


Fig 1.5: Quadrature Demodulator.

Fig 1.5 from the lecture notes shows the quadrature demodulator which performs the specified multiplications and recovers the real and imaginary parts of $p(t)$.

Let $p(t) = i(t) + j\,q(t)$. Then

$$s(t) = Re[p(t)\,e^{j\omega_C t}] = Re[\{i(t) + jq(t)\}\,e^{j\omega_C t}] = i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t)$$

From multiplier 1:

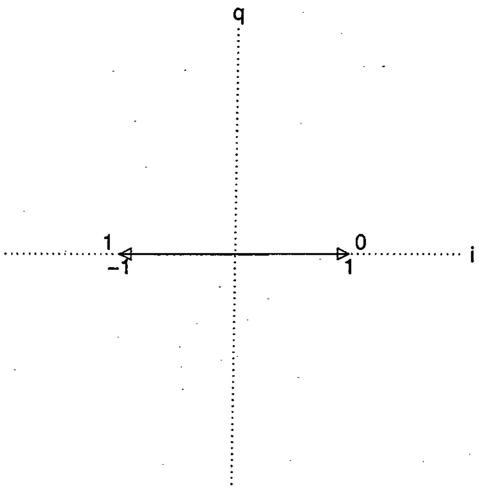
$$\begin{aligned} i'(t) &= s(t) \times 2\cos(\omega_C t) \\ &= [i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t)] \times 2\cos(\omega_C t) \\ &= 2i(t)\cos^2(\omega_C t) - 2q(t)\sin(\omega_C t)\cos(\omega_C t) \\ &= i(t) + i(t)\cos(2\omega_C t) - q(t)\sin(2\omega_C t) \end{aligned}$$

Similarly from multiplier 2:

$$\begin{aligned} q'(t) &= s(t) \times [-2\sin(\omega_C t)] \\ &= [i(t)\cos(\omega_C t) - q(t)\sin(\omega_C t)] \times [-2\sin(\omega_C t)] \\ &= -2i(t)\cos(\omega_C t)\sin(\omega_C t) + 2q(t)\sin^2(\omega_C t) \\ &= q(t) - q(t)\cos(2\omega_C t) - i(t)\sin(2\omega_C t) \end{aligned}$$

When these signals are passed through lowpass filters 1 and 2, the terms modulated onto carriers at $2\omega_C$ are rejected by the filters, and so the outputs of the lowpass filters are $i(t)$ and $q(t)$, as required. [25%]

(b) (i) The constellations are



BPSK phasor diagram.

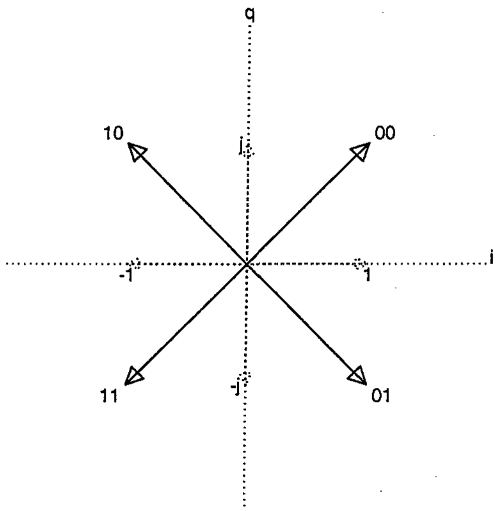


Fig 4.1: QPSK phasor diagram.

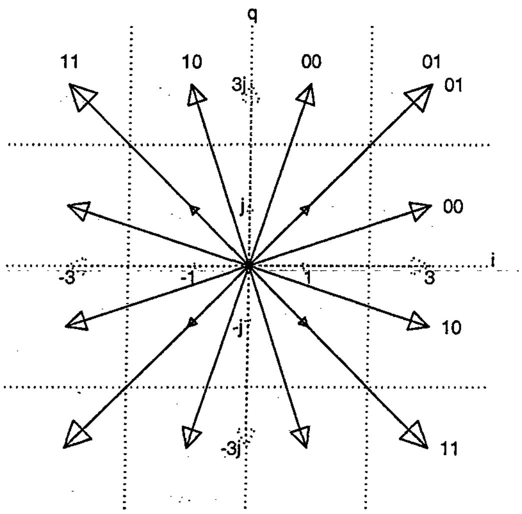


Fig 5.2: 16-QAM phasor diagram

[15%]

(b) (ii) The BER depends approximately on the distance between any pair of adjacent modulation states and on the energy per symbol that is transmitted. QPSK may be regarded as the superposition of two separate BPSK systems in quadrature, which do not interfere with each other, and therefore gives the same BER vs SNR performance as basic BPSK. In both cases

$$P_E = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

16-QAM can be analysed as two separate 4-ASK systems in quadrature. For the same energy per bit, each 4-ASK symbol can have twice the power (mean square value) of the equivalent BPSK symbol (since there are 2 bits in each 4-ASK symbol), but this still leaves adjacent 4-ASK states significantly closer together than in the BPSK case. This results in a loss in performance of approximately 4dB when 16-QAM is compared with BPSK and QPSK. [20%]

(b) (iii) The power spectrum of a data signal, carrying random data, is given by the Fourier Transform of its auto-correlation function (ACF). With all three modulation schemes given, each symbol is uncorrelated with adjacent sybols, and so the ACF is just proportional to the ACF of the shaping pulse for each symbol. In the case of a rectangular shaping pulse, the ACF is a triangular function, and, from the E&I Data Book, the power spectrum is proportional to $\text{sinc}^2(\omega T_s/2)$. Thus the bandwidth is proportional to $1/T_s = 1/(mT_b)$, where m is the number of bits per modulation symbol.

The sinc^2 function is -3.9dB down when its argument = $\pi/2$ (using $10\log_{10}$ since it is a power spectrum), so the -3.9dB bandwidth of the modulated signal is from $\omega = -\pi/T_s$ to $+\pi/T_s$, i.e. a total bandwidth of $f = 1/T_s$ Hz. Hence the -3.9dB bandwidths are:

BPSK: $1/T_b$ Hz

QPSK: $1/(2T_B)$ Hz

16-QAM: $1/(4T_b)$ Hz

The widths of the main lobes of the spectra to the first zeros are twice these values. [20%]

(c) We see from the above results that 16-QAM requires the least bandwidth for a given transmitted bit rate, but it needs about 4dB more signal power than the other two schemes. Hence we would choose 16-QAM (or even more levels such as 64-QAM or 256-QAM) if bandwidth is the most important parameter.

However if error performance in noise is most important, then QPSK is the best choice, since it is joint best with BPSK on error performance and yet is still 2:1 better than BPSK on bandwidth.

Some additional improvement in error performance can be obtained by using a multi-level FSK scheme at the expense of further increases in bandwidth, but this is beyond the scope of the question. [20%]

4 (a) Different multipath delays will cause excessive ISI if the transmitted symbol period is comparable with or less than the path delay differences. Multipath with a delay difference between paths of Δt also tends to cause frequency selective fading at intervals of $1/\Delta t$ across the frequency band, so that a small (but unpredictable) proportion of the frequency band may be unusable.

With a bit rate of 2.46 Mbit/s and QPSK modulation, the transmitted symbol rate is 1.23 Msymbol/s, and so path delay differences of more than 0.8 microseconds would cause severe ISI. The given difference of up to 240 microseconds is clearly way above this limit. Furthermore frequency selective fading would occur at intervals down to about 4 kHz across the signal spectrum. [20%]

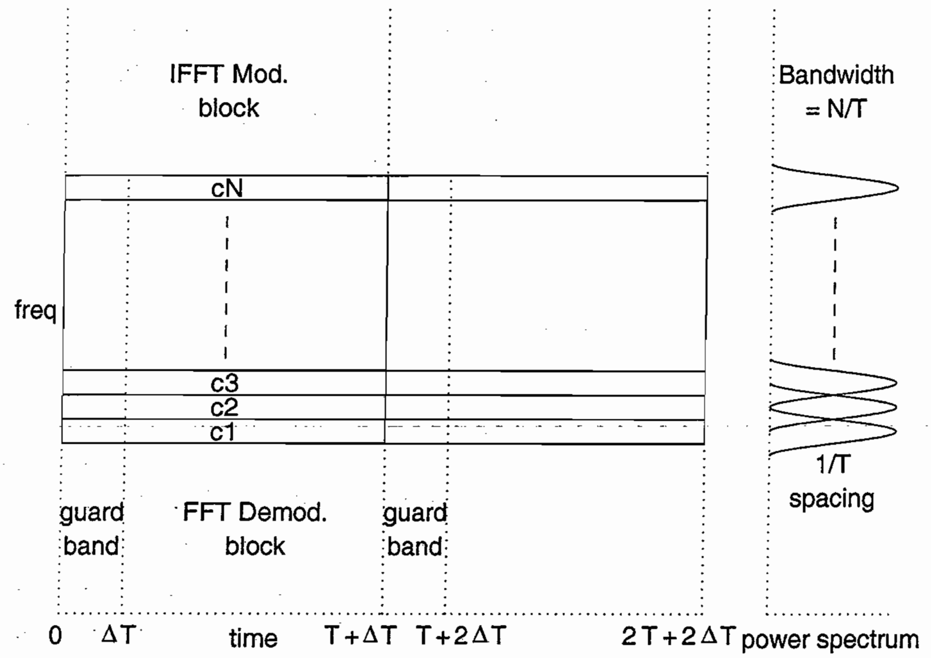


Fig 6.2: Orthogonal Frequency Division Multiplexing (OFDM) with N carriers.

(b) Orthogonal frequency-division multiplexing (OFDM) modulation uses many carriers in parallel to reduce the signaling rate on each carrier. Guard intervals provide minimal degradation due to multipath delay differences.

The aim of OFDM is to demultiplex the high-speed bit stream into N *streams*, each at $1/N$ of the original rate, which are then modulated onto N separate carrier waves, as shown in fig. 6.2 from the lecture notes. These have much improved resilience to typical multipath delays because of their *lower modulation rates*. Typically $N \approx 1000$ to 2000.

The inverse FFT may be used to put QPSK (or QAM) data on each of N carriers, spaced by $1/T$ Hz, where T is the IFFT block period. Each carrier is an IFFT basis function which is multiplied by its Fourier coefficient, the modulation phasor. In this way the *carriers are orthogonal* to each other and may be demodulated by an equivalent FFT process without mutual interference at the receiver. The mutual orthogonality of the IFFT basis functions, means that there should be no interference between each modulated carrier and its neighbours. Orthogonality is not affected by the modulation process, because the modulation rate is no faster than once per FFT block period, so each modulated carrier is a pure tone for the duration of the block period T.

Multipath delays tend to vary across the band of N carriers, and this could upset the orthogonality property because some modulation transitions would need to occur during the demodulator FFT analysis block if each block followed its predecessor immediately. To avoid this problem, a guard period ΔT is inserted between consecutive blocks in the modulator and demodulator. For optimum demodulator performance, the modulator should extend (by periodic extension) the inverse FFT output waveform into the guard period before each block, so that the transmitted waveform is continuous from the point where the modulation transitions occur at the start of each guard period. The FFT demodulator analyses the interval from ΔT to $T + \Delta T$. In this way multipath delays varying from 0 to ΔT can be tolerated without any modulation transitions intruding into the FFT analysis interval and spoiling the orthogonality of the carrier waves. Unfortunately the guard periods either reduce the throughput of the system or increase its bandwidth in the ratio $T : (T + \Delta T)$. [35%]

(c) If the FFT analysis period is 1 ms and the guard period is 0.246 ms, then the total interval between modulation symbols is $1 + 0.246 = 1.246$ ms, giving a symbol rate on each carrier of $1/0.001246 = 802.57$ sym/s.

QPSK modulation encodes 2 bits per symbol, and so with 1536 carriers, the total bit rate available is:

$$1536 \times 2 \times 802.57 = 2.4655 \text{ Mbit/s}$$

which is sufficient to support the specified bit rate.

Since the FFT analysis period is 1 ms, the carrier spacing is $1/0.001 = 1000$ Hz.

Hence the bandwidth of 1536 carriers with this spacing is

$$1536 \times 1000 = 1.536 \text{ MHz}$$

In practice, we need to allow for the sinc^2 shaping of the power spectra of each carrier at the edges of the band, so an additional gap of about 200 kHz (0.2 MHz) is provided between separate DAB blocks. The bandwidth of the complete OFDM signal is therefore approximately

$$1.536 + 0.2 = 1.736 \text{ MHz}$$

[25%]

(d) At the start of the question, it was stated that typically 6 audio signals comprise a DAB block with a user data rate of 1.23 Mbit/s. (This allows about 200 kbit/s per audio channel, which is sufficient for quite good quality stereo with current compression techniques.)

Hence the bandwidth needed per audio channel is

$$1.736 \,.\, 10^6/6 = 289.3 \text{ kHz}$$

This is thus very similar to the 300 kHz of bandwidth needed for an analogue FM signal. So the spectral efficiency seems about the same, and we have just gained the advantage of low noise and clarity of digital transmission.

The big spectral advantage of DAB comes when one considers how to cover a large geographical area. The basic problem is that VHF radio transmissions can only cover ranges of 50 to 100 km (due to the curvature of the Earth) so many transmitters are needed to cover the UK.

In the case of analogue transmissions, it is necessary to allocate separate frequencies to adjacent transmitters so that receivers that are almost equidistant from two transmitters do not receive both signals together on the same channel. In theory one could do this with only 3 separate frequencies per channel, but in practice it is found that 7 frequencies are needed to provide adequately low levels of interference on analogue transmissions.

In the case of DAB signals, the coded OFDM format is designed to tolerate multipath reception with delay differences up to those experienced from transmitters in adjacent regions, so a single frequency allocation per channel is all that is needed to cover an arbitrarily large area. Hence the DAB system ends up being about 7 times more spectrally efficient than the analogue FM system for national broadcasts. [20%]

[Note: these answers are somewhat more detailed than would be expected in an exam.]

Engineering Triops Part 2A
Module 3F4. Data Transmission, May 2007- Answers

1. Attempted by all candidates and was generally well answered. The most common mistakes were to incorrectly calculate the eye height or to neglect to calculate the noise power after the equaliser when evaluating the equalised Bit Error Rate (BER).
a) See notes.
b)

$$H_E(z) = \frac{1}{P(z)} = \frac{1}{1+0.5z^{-1}}$$

Stability not guaranteed.
Numerical instability.
Adaptive methods hard to derive.
c)
(i) 1, -0.5, 0.25, -0.125
(ii) Without equaliser, BER = Q (1.25) = 0.1058. With equaliser, BER = Q (2.03) = 0.0212.

2. This was the second most popular question and was in general answered quite well. A surprising number of candidates could not correctly perform the BER calculation in part (c).
a) (i) and (ii) see notes.
b)
(i) See notes.
(ii)

$$\mathrm{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \mathrm{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

c) $2.923*10^{-3}$

3. This question was in general very well answered. Surprisingly, part (a) proved to be troublesome for some of the candidates even though the solution is in the course notes.
a)

$$i'(t) = i(t) + i(t)\cos(2\omega_C t) - q(t)\sin(2\omega_C t)$$

$$q'(t) = q(t) - q(t)\cos(2\omega_C t) - i(t)\sin(2\omega_C t)$$

Low pass filtering is required to reject the high frequency terms.
b) (i) and (ii) see notes.
(iii) BPSK: $1/T_b$ Hz
QPSK: $1/(2T_b)$ Hz
16-QAM: $1/(4T_b)$ Hz
The widths of the main lobes of the spectra to the first zeros are twice these values.
c) See notes

4. This question was the least popular and also the least well answered. The descriptions of OFDM required in part (b) proved to be quite variable in detail and accuracy. In addition, part (d) proved to be problematic for a number of candidates.
a) See notes.
b) See notes.
c) 802.57 sym/s, total bit rate = 2.4655 Mbit/s, bandwidth = 1.536 MHz, including frequency guard bands gives a total bandwidth = 1.736 MHz, yielding a required bandwidth per audio channel of 289.3 kHz.
d) See notes.