ENGINEERING TRIPOS PART IIA

Tuesday 8 May 2007 9 to 10.30

Module 3C5

DYNAMICS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
Datasheet S32: 3C5 Dynamics and 3C6 Vibration (5 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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1 A symmetrical rotor of mass m shown in Fig. 1 has principal moments of intertia AAC about axes passing through the fixed pivot O. The distance from O to the centre of mass G of the rotor is a. The rotor is spinning at a steady rate ω and is held with its axis vertical, as shown in Fig. 1(a). It is known that if $\omega < \omega_{crit}$, where $\omega_{crit}^2 = \dfrac{4mgaA}{C^2}$, then the rotor will not remain stably upright when it is released.

In a particular experiment the slow-spinning rotor is spining with $\omega = \omega_{crit}/2$. When released the rotor inclination angle θ increases until it reaches a maximum value θ_{max} as shown in Fig. 1(b). The angle θ then returns towards zero. Thereafter oscillations continue, a motion similar to that observed of the gyro-pendulum in the 3C5 laboratory, finally dying out to give steady precession at $\theta = \theta_{final}$ as shown in Fig. 1(c).

(a) Show that the rate of spin ω is constant throughout the motion. [10%]

(b) Show that moment of momentum of the rotor about the vertical axis through O remains constant at a value of $C\omega$ throughout the motion. [10%]

(c) Show that the angle $\theta_{final} = \dfrac{\pi}{2}$ and find the final precession rate. [40%]

(d) Find θ_{max} assuming mechanical energy is conserved during early motion. [40%]

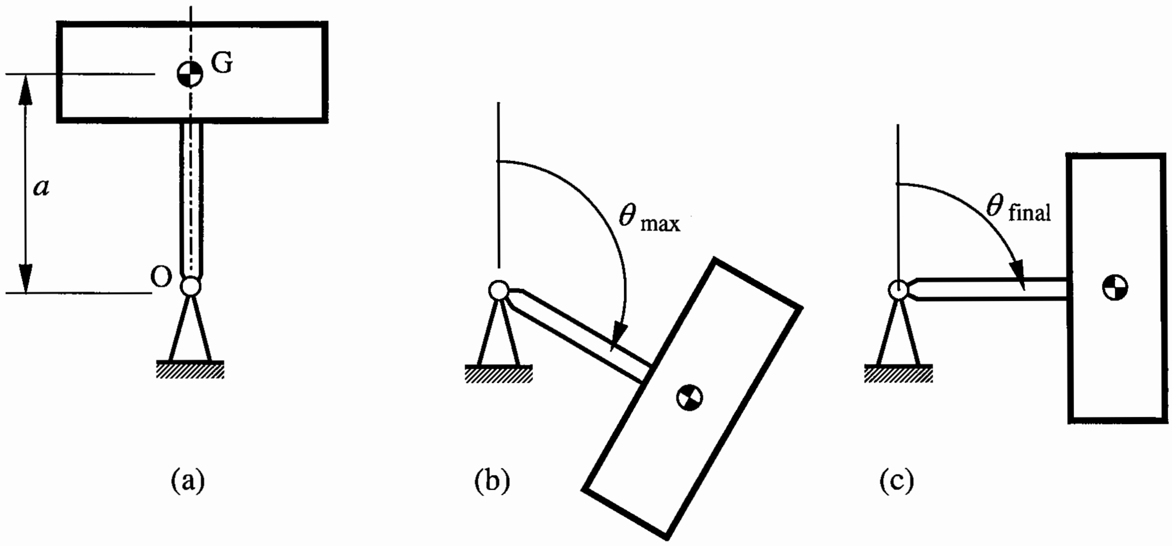

Fig. 1

2 A solid uniform rotor of mass m has a square cross section of side a and has length L. A light rigid shaft passes through the centre of the rotor. The angle between the shaft and the long axis of the rotor is θ as shown in Fig. 2. The shaft is constrained to rotate in fixed bearings at a constant angular velocity Ω.

(a) Find the principal moments of inertia of the rotor. [10%]

(b) Find the components of angular velocity in a reference frame $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ aligned with the principal axes of the rotor and hence find the moment of momentum of the rotor both

(i) in the frame $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$

and (ii) in a reference frame $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$ aligned with $\boldsymbol{K}$ parallel to the shaft.

Use this result to find one of the products of inertia for the rotor in frame $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$. [50%]

(c) Find an expression for the magnitude of the couple Q acting on the shaft through its bearings. [30%]

(d) For what value of L is the value of Q equal to zero? [10%]

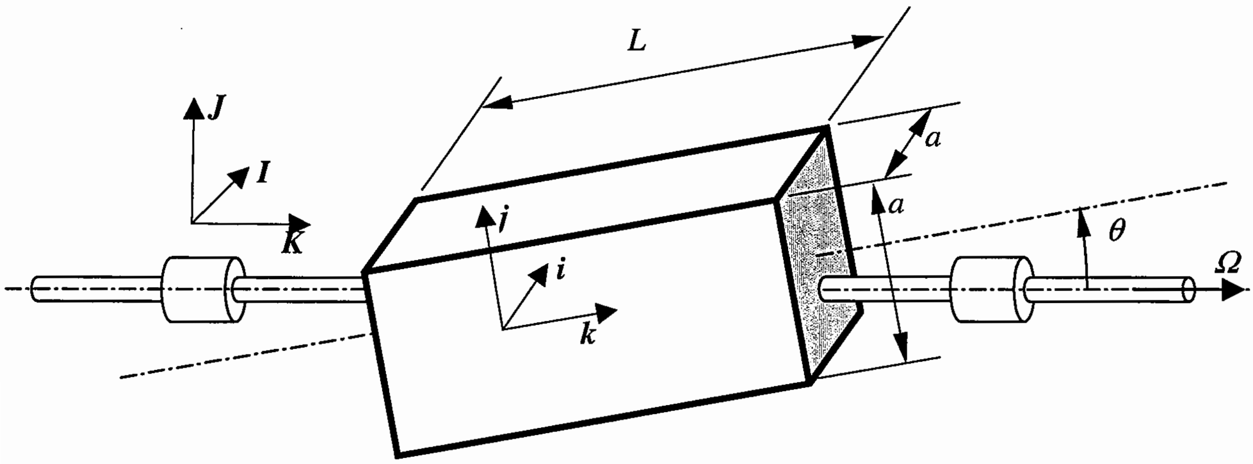

Fig. 2

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3 A thin coin of mass m and radius a is wobbling on a horizontal table as shown in Fig. 3. The motion can be analysed by assuming it to be in steady state with the centre of the coin G at rest. The coin is in contact with the table at point P where it rolls without slip. The motion of the coin is described by Euler angles θ , ϕ and ψ. Unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (not fixed in the body) are defined in the figure. The unit vector $\boldsymbol{j}$ is always horizontal.

(a) Draw a free-body diagram of the coin and hence find a vector expression for the couple acting on the coin. [10%]

(b) Find $\dot{\phi}$, the steady rate of wobbling of the coin. [50%]

(c) Find an expression for the rate of turning of the head of the coin as viewed from above. Make suitable approximations for small θ. [40%]

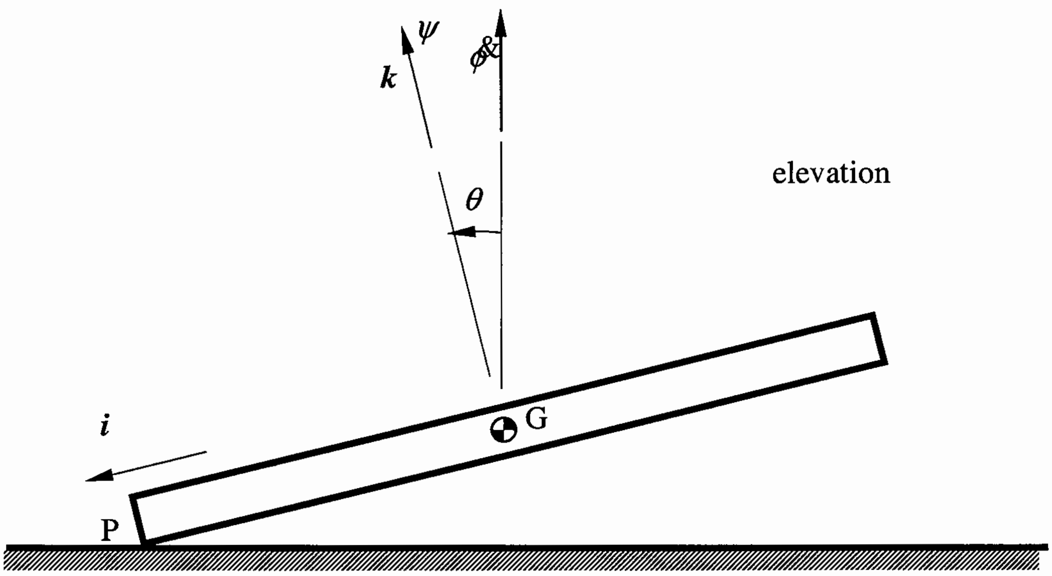

Fig. 3

4 A two degree of freedom system shown in Fig. 4 comprises a simple pendulum of mass m_2 and length L connected to a mass on a spring. The mass-spring system has mass m_1 and spring stiffness k. The system moves in a vertical plane, and the degrees of freedom are taken to be the rotation θ of the pendulum and the vertical displacement x of the mass m_1. The acceleration due to gravity is g.

(a) Show that the kinetic energy of the system is given by

$$T = \frac{1}{2}\left\{(m_1 + m_2)\dot{x}^2 + m_2\left(L^2\dot{\theta}^2 - 2L\dot{x}\dot{\theta}\sin\theta\right)\right\}$$

and find also an expression for the potential energy of the system. [20%]

(b) By using Lagrange's equation derive the equations of motion of the system. Do *not* assume that the rotation θ is small at this stage. [40%]

(c) Assume now that both θ and x are small and hence find from your answers to (b) the linear equations of motion which govern small amplitude free vibrations of the system. [20%]

(d) Show that the same equations can be derived directly from part (a). [20%]

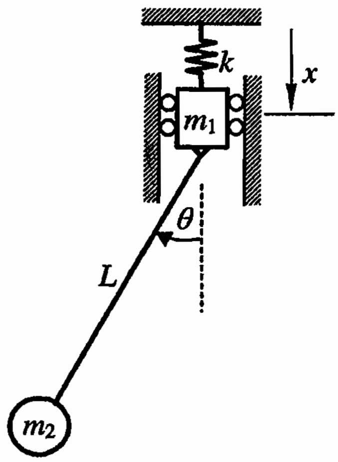

Fig. 4

5 An articulated tower of length L is used as an offshore mooring terminal as shown in Fig. 5. The tower is pivoted at the seabed and has moment of inertia I about the pivot. The combined effect of gravity and buoyancy is to produce a constant upwards force P acting at the centre of the tower. Incoming waves produce a horizontal time varying force $F(t)$ acting at the centre of the tower. The tower moves in the vertical plane and has a single degree of freedom which can be taken to be either the rotation of the tower θ or the horizontal displacement of the top of the tower x.

(a) Taking the degree of freedom to be x, show that the kinetic energy T of the tower and the generalized force Q (due to P and F) are given by

$$T = \frac{I\dot{x}^2}{2(L^2 - x^2)} \qquad Q = -\frac{xP}{2\sqrt{L^2 - x^2}} + \frac{F}{2}$$

and explain why the potential energy $V = 0$ throughout the motion. [30%]

(b) By using Lagrange's equation derive the equation of motion that governs x. Hence show that the natural frequency of small oscillations of the tower is $\sqrt{LP/2I}$. [30%]

(c) Now take the degree of freedom to be θ. Find new expressions for T and Q. Use Lagrange's equation again to find the equation that governs θ. Show that this can be derived from the equation for x by making the substitution $x = L\sin\theta$. [40%]

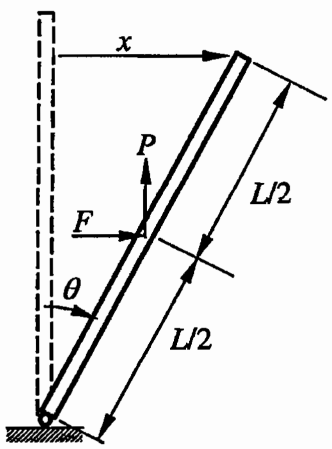

Fig. 5

END OF PAPER

Part IIA Data sheet
Module 3C5 Dynamics
Module 3C6 Vibration

Dynamics in three dimensions

Axes fixed in direction

(a) Linear momentum for a general collection of particles m_i:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^{(e)}$$

where $\boldsymbol{p} = M\,\boldsymbol{v}_G$, M is the total mass, $\boldsymbol{v}_G$ is the velocity of the centre of mass and $\boldsymbol{F}^{(e)}$ the total external force applied to the system.

(b) Moment of momentum about a general point P

$$\boldsymbol{Q}^{(e)} = (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{h}}_G$$
$$= \dot{\boldsymbol{h}}_P + \dot{\boldsymbol{r}}_P \times \boldsymbol{p}$$

where $\boldsymbol{Q}^{(e)}$ is the total moment of external forces about P. Here, $\boldsymbol{h}_P$ and $\boldsymbol{h}_G$ are the moments of momentum about P and G respectively, so that for example

$$\boldsymbol{h}_P = \sum_i (\boldsymbol{r}_i - \boldsymbol{r}_P) \times m_i \dot{\boldsymbol{r}}_i$$
$$= \boldsymbol{h}_G + (\boldsymbol{r}_G - \boldsymbol{r}_P) \times \boldsymbol{p}$$

where the summation is over all the mass particles making up the system.

(c) For a rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed point P at the origin of coordinates

$$\boldsymbol{h}_P = \int \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})\, dm = \boldsymbol{I}\,\boldsymbol{\omega}$$

where the integral is taken over the volume of the body, and where

$$\boldsymbol{I} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and $A = \int (y^2 + z^2)dm \qquad B = \int (z^2 + x^2)dm \qquad C = \int (x^2 + y^2)dm$

$D = \int yz\, dm \qquad E = \int zx\, dm \qquad F = \int xy\, dm$

where all integrals are taken over the volume of the body.

Axes rotating with angular velocity Ω

Time derivatives of vectors must be replaced by the "rotating frame" form, so that for example

$$\dot{\boldsymbol{p}} + \boldsymbol{\Omega} \times \boldsymbol{p} = \boldsymbol{F}^{(e)}$$

where the time derivative is evaluated in the moving reference frame.

When the rate of change of the position vector $\boldsymbol{r}$ is needed, as in 1(b) above, it is usually easiest to calculate velocity components directly in the required directions of the axes. Application of the general formula needs an extra term unless the origin of the frame is fixed.

Euler's dynamic equations (governing the angular motion of a rigid body)

(a) Body-fixed reference frame:

$$A\,\dot{\omega}_1 - (B-C)\,\omega_2\,\omega_3 = Q_1$$
$$B\,\dot{\omega}_2 - (C-A)\,\omega_3\,\omega_1 = Q_2$$
$$C\,\dot{\omega}_3 - (A-B)\,\omega_1\,\omega_2 = Q_3$$

where A, B and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes aligned with the principal axes of inertia of the body at P.

(b) Non-body-fixed reference frame for axisymmetric bodies (the "Gyroscope equations"):

$$A\,\dot{\Omega}_1 - (A\,\Omega_3 - C\,\omega_3)\,\Omega_2 = Q_1$$
$$A\,\dot{\Omega}_2 + (A\,\Omega_3 - C\,\omega_3)\,\Omega_1 = Q_2$$
$$C\,\dot{\omega}_3 = Q_3$$

where A, A and C are the principal moments of inertia about P which is either at a fixed point or at the centre of mass. The angular velocity of the body is $\omega = [\omega_1, \omega_2, \omega_3]$ and the moment about P of external forces is $Q = [Q_1, Q_2, Q_3]$ using axes such that ω_3 and Q_3 are aligned with the symmetry axis of the body. The reference frame (not fixed in the body) rotates with angular velocity $\Omega = [\Omega_1, \Omega_2, \Omega_3]$ with $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$.

Lagrange's equations

For a holonomic system with generalised coordinates q_i

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

where T is the total kinetic energy, V is the total potential energy, and Q_i are the non-conservative generalised forces.

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{q}^t K \underline{q}}{\underline{q}^t M \underline{q}}$ where $\underline{q}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p5.

If this quantity is evaluated with any vector $\underline{q}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to $\omega_k{}^2$ if $\underline{q}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

VIBRATION MODES AND RESPONSE

Discrete systems

1. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n^{\ 2} M\underline{u}^{(n)}$$

where the M (mass matrix) and K (stiffness matrix) are both symmetric and positive definite.

Continuous systems

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p5) and boundary conditions, assuming harmonic time dependence.

2. Kinetic energy

Discrete:

$$T = \frac{1}{2}\underline{\dot{u}}^t M \underline{\dot{u}}$$

Continuous:

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

3. Orthogonality and normalisation

Discrete:

$$\underline{u}^{(j)^t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)^t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

Continuous:

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

4. General response

Discrete: The general response of the system can be written as a sum of modal responses

$$\underline{q}(t) = \sum_n a_n(t)\, \underline{u}^{(n)}$$

where $\underline{q}$ is the vector of generalised coordinates and a_n gives the "amount" of the nth mode.

Continuous: The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_n a_n(t)\, u_n(x)$$

where $w(x,t)$ is the displacement and a_n gives the "amount" of the nth mode.

5. Transfer function

Discrete: For (generalised) force F at frequency ω, applied at point (or generalised coordinate) j, and response q measured at point (or generalised coordinate) k the transfer function is

$$H(j,k,\omega) = \frac{q}{F} = \sum_n \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

Continuous: For force F at frequency ω applied at point x, and response w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^{\ 2} - \omega^2}$$

(with no damping), or

$$H(j,k,\omega)=\frac{q}{F}\approx\sum_n\frac{u_j^{(n)}u_k^{(n)}}{\omega_n^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

$$H(x,y,\omega)=\frac{w}{F}\approx\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n^2+2i\omega\omega_n\zeta_n-\omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

6. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)}u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\,u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

7. Impulse response

For a unit impulse applied at $t=0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$g(j,k,t)=\sum_n\frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t$$

(with no damping), or

$$g(j,k,t)\approx\sum_n\frac{u_j^{(n)}u_k^{(n)}}{\omega_n}\sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

(with small damping).

For a unit impulse applied at $t=0$ at point x, the response at point y is

$$g(x,y,t)=\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t$$

(with no damping), or

$$g(x,y,t)\approx\sum_n\frac{u_n(x)\,u_n(y)}{\omega_n}\sin\omega_n t\, e^{-\omega_n\zeta_n t}$$

(with small damping).

8. Step response

For a unit step force applied at $t=0$ at point (or generalised coordinate) j, the response at point (or generalised coordinate) k is

$$h(j,k,t)=\sum_n u_j^{(n)}u_k^{(n)}\left[1-\cos\omega_n t\right]$$

(with no damping), or

$$h(j,k,t)\approx\sum_n u_j^{(n)}u_k^{(n)}\left[1-\cos\omega_n t\, e^{-\omega_n\zeta_n t}\right]$$

(with small damping).

For a unit step force applied at $t=0$ at point x, the response at point y is

$$h(x,y,t)=\sum_n u_n(x)\,u_n(y)\left[1-\cos\omega_n t\right]$$

(with no damping), or

$$h(t)\approx\sum_n u_n(x)\,u_n(y)\left[1-\cos\omega_n t\, e^{-\omega_n\zeta_n t}\right]$$

(with small damping).

Governing equations for continuous systems

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}P\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}m\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)\left(a^4 - b^4\right)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\frac{\partial^2 \theta}{\partial t^2} - GJ\frac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}GJ\int\left(\frac{\partial \theta}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho J\int\left(\frac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} - EA\frac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \frac{1}{2}EA\int\left(\frac{\partial w}{\partial x}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \frac{1}{2}EI\int\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \frac{1}{2}\rho A\int\left(\frac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

Engineering Tripos Part IIA 2007

Answers

1. (c) $\dot{\phi} = \frac{mga}{C\omega} = \sqrt{\frac{mga}{A}} = \frac{C\omega}{A}$ (d) 120°

2. (a) $m(a^2+L^2)/12$, $m(a^2+L^2)/12$, $ma^2/6$
   (b) $\boldsymbol{\omega} = 0\,\boldsymbol{i} - \Omega\sin\theta\,\boldsymbol{j} + \Omega\cos\theta\,\boldsymbol{k}$
   (i) $\boldsymbol{h} = -\mathrm{A}\Omega\sin\theta\,\boldsymbol{j} + \mathrm{C}\Omega\cos\theta\,\boldsymbol{k}$
   (ii) $\boldsymbol{h} = (C-A)\Omega\cos\theta\sin\theta\,\boldsymbol{J} + (A\Omega\sin^2\theta + C\Omega\cos^2\theta)\,\boldsymbol{K}$
   product of inertia $= -(C-A)\cos\theta\sin\theta$
   (c) $Q_1 = (A-C)\Omega^2\cos\theta\sin\theta$
   (d) $L = a$

3. (a) $\boldsymbol{Q} = -\,mga\cos\theta\,\boldsymbol{j}$ (b) $2\sqrt{\frac{g}{a\tan\theta}}$ (c) $\sqrt{\frac{g\theta^3}{a}}$

4. (a) $V = \frac{1}{2}k\theta^2 - m_1 gx - m_2 g(x - L(1-\cos\theta))$
   (b) $(m_1+m_2)\ddot{x} - m_2 L\ddot{\theta}\sin\theta - m_2 L\dot{\theta}^2\sin\theta\cos\theta + kx - (m_1+m_2)g = 0$
   $L\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0$
   (c) $(m_1+m_2)\ddot{x} + kx = (m_1+m_2)g$
   $L\ddot{\theta} + g\theta = 0$
   (d) $M = \begin{bmatrix} m_1+m_2 & 0 \\ 0 & m_2L^2 \end{bmatrix}$ $K = \begin{bmatrix} k & 0 \\ 0 & m_2gL \end{bmatrix}$

5. (b) $\frac{I\ddot{x}}{L^2-x^2} + \frac{I\dot{x}^2x}{(L^2-x^2)^2} = \frac{F}{2} - \frac{Px}{2\sqrt{L^2-x^2}}$
   (c) $T = \frac{1}{2}I\dot{\theta}^2$ $Q = \frac{FL\cos\theta}{2} - \frac{PL\sin\theta}{2}$

HEMH May 2007