ENGINEERING TRIPOS   PART IIA

Thursday 3 May 2007   9 to 10.30

Module 3C7

MECHANICS OF SOLIDS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
3C7 datasheet

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
Graph paper	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A square plate ABCD undergoes a plane strain to ABC*D, as shown in Fig. 1, in such a way that points along the x- and y- axes remain fixed and such that the component v of the displacement in the y-direction of any point $P(x,y)$ is $v(x,y) = 0$. The displacement in the x-direction is $u(x,y) = (\lambda xy)/L^2$, where L is the edge length of the plate and λ is a constant satisfying $|\lambda / L| << 1$. The plate is elastic with a Young's modulus E and a Poisson's ratio μ.

(a) Plot Mohr's circle of strain for an arbitrary point $P(x,y)$, and hence find the principal strains, maximum shear strain and the corresponding directions. [30%]

(b) Determine the extension Δ_{AC} of the line AC, and hence find the resulting average strain $\bar{\varepsilon}_{AC}$ of this diagonal, AC. Make use of the Binomial expansion for $(1 + x)^n$ when $x<<1$). [30%]

(c) Determine the stress field in the plate. Sketch the normal and shear stresses acting on all four edges and determine if these edge-loads satisfy the conditions of overall equilibrium. [40%]

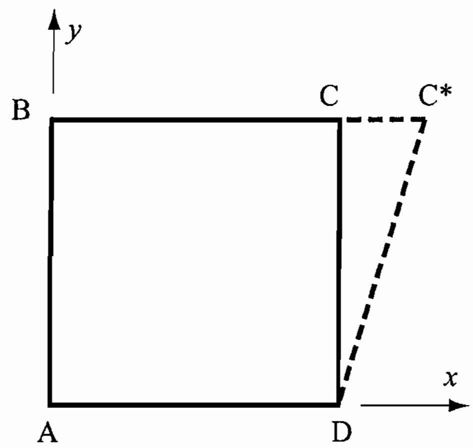

Fig. 1

2 (a) A shaft, which has an equilateral triangular cross-section, is subject to axial torque T and develops twist α about its longitudinal axis. The coordinates x,y originate at one corner of the cross-section, as shown in Fig. 2 and the centroid is at C. The shear modulus of the material is G.

(a) Find the coefficient η such that $\phi(x,y)$ shown below is a Prandtl function for the state of stress in the shaft

$$\phi = -\frac{G\alpha}{2}\left(\eta y^2 - x^2\right)\left(1 - \frac{x}{a}\right).$$ [30%]

(b) Show that the torsional stiffness of the shaft T/α can be expressed as

$$\frac{T}{\alpha} = \frac{\sqrt{3}a^4 G}{45}.$$ [30%]

(c) Identify where the maximum shear stress occurs and obtain an expression for its value under the specified loading conditions. [30%]

(d) Without doing further calculations, state where the shear stress is zero and explain why. [10%]

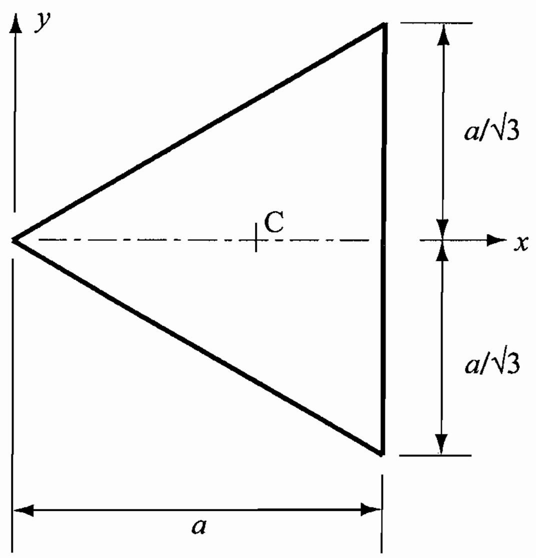

Fig. 2

(TURN OVER

3 High-quality surface finishes can be produced on plane metallic surfaces by passing the work-piece under a rigid tool. Figure 3 shows the softer work-piece A (unit depth into the plane of the paper), which is being drawn at a speed v from right to left, by a force F, under the rigid tool which is loaded by a force N. At steady-state, the tip of the tool is at the same level as the incoming free surface of the work-piece with a "wave" of plastically deforming material (regions B and C) at the leading edge of the tool. The angles between the various regions are as indicated in Fig. 3 and the length of contact between the tool D and block C is l.

(a) Find the length, in terms of l, of the interface between regions B and C. [20%]

(b) Draw, to scale, a velocity diagram for this mode of deformation. [20%]

(c) Estimate the value of the force F in terms of the shear yield strength of the material k and the length l. It can be assumed that there is no friction between the tool and the work-piece. Is your estimate an upper or lower bound? [40%]

(d) By considering the equilibrium of the tool, obtain the relationship between l and the applied load N. [20%]

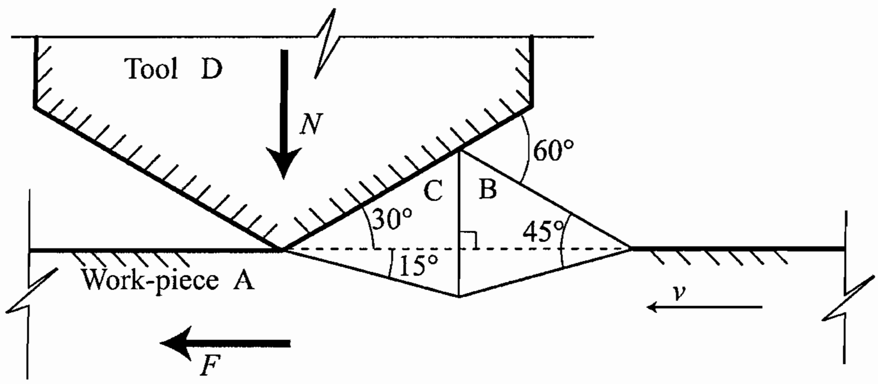

Fig. 3

4 (a) Show that

$$\phi(r,\theta) = C_1 \sin 2\theta + C_2\theta$$

is a valid Airy stress function in polar coordinates. [15%]

(b) Figure 4 is an elevation of a tapering beam, of unit width into the plane of the paper, that is loaded at each end by a moment M. It has been proposed that the stress function from (a) can be used to determine the stress field in the beam.

(i) Write down the stress boundary conditions to be satisfied on the top and bottom faces. [15%]

(ii) Show that stresses derived from $\phi(r,\theta)$ satisfy these boundary conditions and hence determine the relation between C_1 and C_2. [20%]

(c) For the applied moment M determine the stresses in the beam as a function of r and θ. [30%]

(d) Show that, for a long thin beam (where $(1-a/b) >> \alpha$), the stresses σ_{rr} are those that would be expected from simple beam theory. [20%]

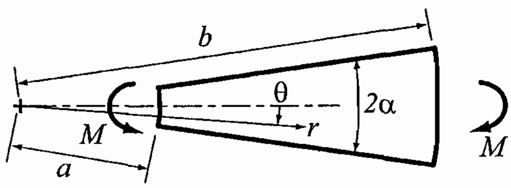

Fig. 4

END OF PAPER

Paper 3C7: Mechanics of Solids

ELASTICITY and PLASTICITY FORMULAE

1. Axi-symmetric deformation : discs, tubes and spheres

	Discs and tubes	Spheres
Equilibrium	$\sigma_{\theta\theta} = \dfrac{d(r\sigma_{rr})}{dr} + \rho\omega^2 r^2$	$\sigma_{\theta\theta} = \dfrac{1}{2r}\dfrac{d(r^2\sigma_{rr})}{dr}$
Lamé's equations (in elasticity)	$\sigma_{rr} = A - \dfrac{B}{r^2} - \dfrac{3+\nu}{8}\rho\omega^2 r^2 - \dfrac{E\alpha}{r^2}\int_c^r rT\,dr$	$\sigma_{rr} = A - \dfrac{B}{r^3}$
	$\sigma_{\theta\theta} = A + \dfrac{B}{r^2} - \dfrac{1+3\nu}{8}\rho\omega^2 r^2 + \dfrac{E\alpha}{r^2}\int_c^r rT\,dr - E\alpha T$	$\sigma_{\theta\theta} = A + \dfrac{B}{2r^3}$

2. Plane stress and plane strain

	Cartesian coordinates	Polar coordinates
Strains	$\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$	$\varepsilon_{rr} = \dfrac{\partial u}{\partial r}$
	$\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$	$\varepsilon_{\theta\theta} = \dfrac{u}{r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$
	$\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$	$\gamma_{r\theta} = \dfrac{\partial v}{\partial r} + \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r}$
Compatibility	$\dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2}$	$\dfrac{\partial}{\partial r}\left\{r\dfrac{\partial \gamma_{r\theta}}{\partial \theta}\right\} = \dfrac{\partial}{\partial r}\left\{r^2\dfrac{\partial \varepsilon_{\theta\theta}}{\partial r}\right\} - r\dfrac{\partial \varepsilon_{rr}}{\partial r} + \dfrac{\partial^2 \varepsilon_{rr}}{\partial \theta^2}$
or (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}(\sigma_{xx} + \sigma_{yy}) = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\}(\sigma_{rr} + \sigma_{\theta\theta}) = 0$
Equilibrium	$\dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0$	$\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{\partial \sigma_{r\theta}}{\partial \theta} - \sigma_{\theta\theta} = 0$
	$\dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0$	$\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial}{\partial r}(r\sigma_{r\theta}) + \sigma_{r\theta} = 0$
$\nabla^4\phi = 0$ (in elasticity)	$\left\{\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right\}\left\{\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2}\right\} = 0$	$\left\{\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}\right\} \times \left\{\dfrac{\partial^2 \phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}\right\} = 0$
Airy Stress Function	$\sigma_{xx} = \dfrac{\partial^2 \phi}{\partial y^2}$	$\sigma_{rr} = \dfrac{1}{r}\dfrac{\partial \phi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$
	$\sigma_{yy} = \dfrac{\partial^2 \phi}{\partial x^2}$	$\sigma_{\theta\theta} = \dfrac{\partial^2 \phi}{\partial r^2}$
	$\sigma_{xy} = -\dfrac{\partial^2 \phi}{\partial x \partial y}$	$\sigma_{r\theta} = -\dfrac{\partial}{\partial r}\left\{\dfrac{1}{r}\dfrac{\partial \phi}{d\theta}\right\}$

3. **Torsion of prismatic bars**

Prandtl stress function: $\sigma_{zx}\ (=\tau_x) = \dfrac{dF}{dy}$, $\sigma_{zy}\ (=\tau_y) = -\dfrac{dF}{dx}$

Equilibrium: $T = 2\int_A F\,dA$

Governing equation for elastic torsion: $\nabla^2 F = -2G\beta$ where β is the angle of twist per unit length.

4. **Total potential energy of a body**

$$\Pi = U - W$$

where $U = \dfrac{1}{2}\int_V \underset{\sim}{\varepsilon}^T [D]\, \underset{\sim}{\varepsilon}\, dV$, $W = \underset{\sim}{P}^T \underset{\sim}{u}$ and $[D]$ is the elastic stiffness matrix.

5. **Principal stresses and stress invariants**

Values of the principal stresses, σ_p, can be obtained from the equation

$$\begin{vmatrix} \sigma_{xx}-\sigma_p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy}-\sigma_p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}-\sigma_p \end{vmatrix} = 0$$

This is equivalent to a cubic equation whose roots are the values of the 3 principal stresses, i.e. the possible values of σ_p.

Expanding: $\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$ where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$,

$$I_2 = \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{yz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{vmatrix} \quad \text{and} \quad I_3 = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}.$$

6. **Equivalent stress and strain**

Equivalent stress $\bar{\sigma} = \sqrt{\dfrac{1}{2}}\ \{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\}^{1/2}$

Equivalent strain increment $d\bar{\varepsilon} = \sqrt{\dfrac{2}{3}}\ \{d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2\}^{1/2}$

7. **Yield criteria and flow rules**

Tresca

Material yields when maximum value of $|\sigma_1-\sigma_2|$, $|\sigma_2-\sigma_3|$ or $|\sigma_3-\sigma_1|$ $= Y = 2k$, and then,

if σ_3 is the intermediate stress, $d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \lambda(1 : -1 : 0)$ where $\lambda \neq 0$.

von Mises

Material yields when, $(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6k^2$, and then

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = \frac{d\varepsilon_1 - d\varepsilon_2}{\sigma_1-\sigma_2} = \frac{d\varepsilon_2 - d\varepsilon_3}{\sigma_2-\sigma_3} = \frac{d\varepsilon_3 - d\varepsilon_1}{\sigma_3-\sigma_1} = \lambda = \frac{3}{2}\frac{d\bar{\varepsilon}}{\bar{\sigma}}$$