

ENGINEERING TRIPOS PART IIA

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Friday 27 April 2007

9 to 10.30

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Module 3D2

GEOTECHNICAL ENGINEERING

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachment: Soil Mechanics Data Book (19 pages)*

STATIONERY REQUIREMENTS

Single-sided script paper

Graph paper

SPECIAL REQUIREMENTS

Engineering Data Book

CUED approved calculator allowed

**You may not start to read the questions  
printed on the subsequent pages of this  
question paper until instructed that you  
may do so by the Invigilator**

1 A layer of soft clay on the bank of a river estuary has been formed in the past by deposition of mud below water. A borehole sample is to be taken at a depth of 25 m below the present day ground surface. The ground water level is at the ground surface. The following stress history has been established:

Stage A: In the past the maximum effective vertical stress experienced by the sample was when 40 m of clay existed above it, with the water table coincident with the top of the clay.

Stage B: The top 15 m of clay was then gradually eroded and the water level fell, remaining coincident with the top of the clay.

The unit weight of the clay can be taken to have remained at  $18 \text{ kN/m}^3$  throughout, both above and below the water table. The clay has a critical state angle of friction  $\phi_{crit} = 25^\circ$  and the critical stress ratio in triaxial compression is  $M = 1.0$ .

(a) Using the Soil Mechanics Data Book estimate the coefficient of earth pressure at rest,  $K_0$ , for the sample after each of stages A and B of its stress history. [40%]

(b) Using graph paper, plot in  $q-p'$  space the effective stress states relevant to the sample at the end of each of these two stages of stress history. Sketch the effective stress paths. [20%]

(c) The sample is then extracted from the ground without any disturbance and split into two identical specimens, which are set up in a triaxial test apparatus with the same total and effective stress state as they had experienced in the ground immediately prior to sampling. Undrained triaxial compression tests are then undertaken. In test 1 the axial stress  $\sigma_a$  is increased, while keeping the cell pressure  $\sigma_r$  constant. In test 2 the cell pressure  $\sigma_r$  is reduced, while keeping the axial stress  $\sigma_a$  constant. In both tests the same stress difference  $(\sigma_a - \sigma_r) = 140 \text{ kN/m}^2$  was ultimately observed, in what can be regarded as a critical state. On the same plot you drew for (b), add the effective and total stress states in  $q-p'$  space for each triaxial test and sketch the total and effective stress paths. What pore pressure would have been measured at failure for each test? [40%]

2 A long tunnel of diameter 6 m is constructed in a uniform body of saturated clay with its axis at a depth of 25 m. The clay has a unit weight of  $20 \text{ kN/m}^3$ , an undrained shear strength  $c_u$  of  $100 \text{ kN/m}^2$  constant with depth, and a linear elastic shear modulus of  $10 \text{ MN/m}^2$ . The measured radial ground movement at the tunnel boundary is 20 mm.

(a) By assuming the tunnel construction to be an axisymmetric contracting cavity under undrained conditions, analogous to cylindrical cavity expansion, use the Soil Mechanics Data Book to estimate the average radial stress imposed on the tunnel lining, which should be assumed to be smooth. [40%]

(b) At one location the tunnel passes beneath a deep foundation supporting a sensitive building. The base of the foundation is 6 m above the crown of the tunnel. Ignoring any effects of the loading of the foundation, and assuming the same average radial stress imposed on the lining as you have calculated for (a), estimate the ground settlement at the foundation. [20%]

(c) In the elastic zone of the soil, at any radius  $r$ , the following expressions apply:

$$\sigma_r = \sigma_0 - G\delta A / \pi r^2$$

$$\sigma_\theta = \sigma_0 + G\delta A / \pi r^2$$

where  $\sigma_r$  and  $\sigma_\theta$  are the radial and circumferential stresses respectively,  $\sigma_0$  is the original in situ total stress in the ground,  $G$  is the elastic shear modulus of the soil, and  $\delta A$  is the contraction of the cavity (expressed as a change in its cross-sectional area  $A$ ).

By considering the radius of the elastic/plastic boundary, and making the same assumptions as for (a) and (b), calculate the distance above the tunnel crown at which the foundation would have to be located to ensure that it remained above the plastic zone associated with the tunnel construction. [40%]

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3 It is required to make an approximate calculation of the movement to be expected in a stratum of clay of depth  $D$  when a wide, stiff diaphragm wall of depth  $L$  is pinned by a prop at original ground level prior to the excavation of a depth  $H$  of soil on one side of the wall, as shown in cross-section in Fig. 1a. The clay can be taken to have a mean undrained shear strength  $c_u$  which holds constant for shear strains  $\gamma > \gamma_u$ , and a mobilised shear stress-strain relation  $(\tau / c_u) = (\gamma / \gamma_u)^b$  for shear strains  $\gamma < \gamma_u$ . The bulk density of the clay can be taken to be a constant,  $\rho$ .

It may be assumed that the wall with the adjacent soil rotates as a rigid body about the prop at O, as shown in Fig. 1b, and that the equilibrium and compatibility of the surrounding ground can be inspected in terms of the family of circular arcs centred at O and having radii  $R$  between  $L$  and  $D$ .

(a) Develop an expression for the shear stress  $\tau$  required for equilibrium of the ground lying above an arc of radius  $R$ . Show that this simplifies to  $\tau = \rho g H / 2\pi$  when  $H \ll R$ , and that the approximation  $\tau = \rho g H / 6$  holds within 5% for  $H/R$  between 0 and 0.5 .

[40%]

(b) Using this approximation, derive an expression for the local shear strain  $\gamma$  at radius  $R$ , and use this to estimate the wall rotation  $\delta\omega$  in terms of other parameters.

[30%]

(c) In a particular case of the top-down construction of a subway station, the following parameter values can be used:  $D = 40$  m,  $L = 20$  m, in soil with  $c_u = 40$  kPa,  $\gamma_u = 10\%$ ,  $\rho = 1650$  kg / m<sup>3</sup>, and  $b = 0.5$ . Estimate the depth of excavation  $H$  that will be permissible for such a diaphragm wall before a second row of props must be provided. The required criterion is that the wall rotation during excavation should not exceed 0.002 radians to avoid damage to neighbouring facilities.

[20%]

(d) Discuss briefly the usefulness or otherwise of conventional factors of safety in a case such as this.

[10%]

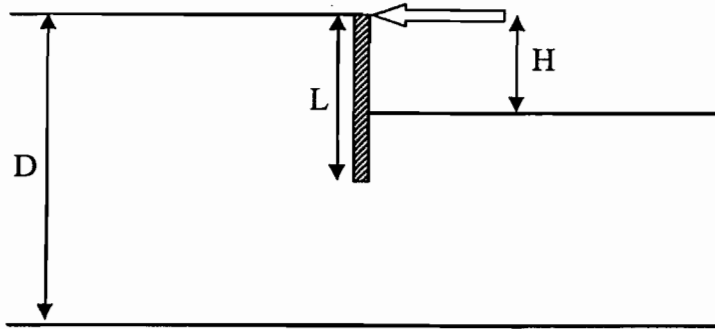


Fig. 1a

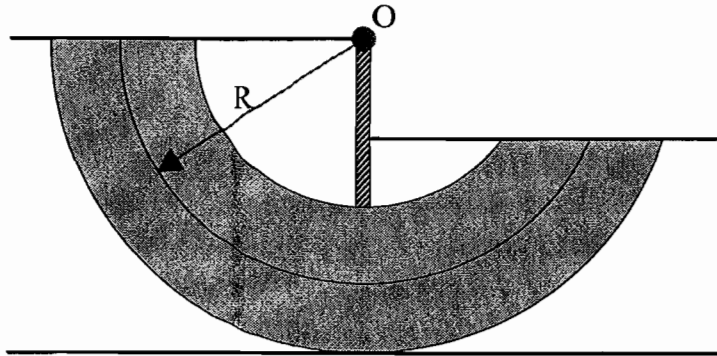


Fig. 1b

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4 The data of Ham River sand given on page 10 of the Soil Mechanics Data Book is sufficient to make alternative predictions of the drained shear strength and dilatancy of a sample with a given history and state path:

A: using Cam Clay yield surfaces and the normality principle,

B: using empirical correlations based on the Relative Dilatancy Index  $I_R$ .

(a) A dry sample of Ham River sand is placed in a Simple Shear Apparatus and compacted by vibration to a minimum voids ratio of 0.60. It is subjected to a constant vertical confining stress of 150 kPa and sheared. Use each of methods A and B in turn to make predictions of the peak strength of the sand during the test, and of the corresponding maximum rate of dilatancy expressed as an angle of dilation  $\psi$ . Comment on the reasons for the discrepancies between the two prediction methods.

[50%]

(b) Neither approach A nor B is sufficient to describe the response of sand to cyclic shear strains. This failure is thought to arise from the inelastic response of soils to stress cycles “inside the Cam Clay yield surface” due to small shear strain cycles. Use the equation of plastic work and dissipation to distinguish the effects of permanent plastic strains arising from such cycling above and below the critical state stress ratio.

[20%]

(c) Describe, explain, and suggest one means of avoiding, the different problems to be expected when:

(i) the shallow foundations of an offshore wind turbine on very dense sand are allowed to mobilise 90% of the peak angle of friction of the sand under extreme but foreseeable storm conditions;

[15%]

(ii) a river crossing over a deep stratum of medium to loose sand, with bridge piers supported on groups of slender piles, experiences a strong earthquake.

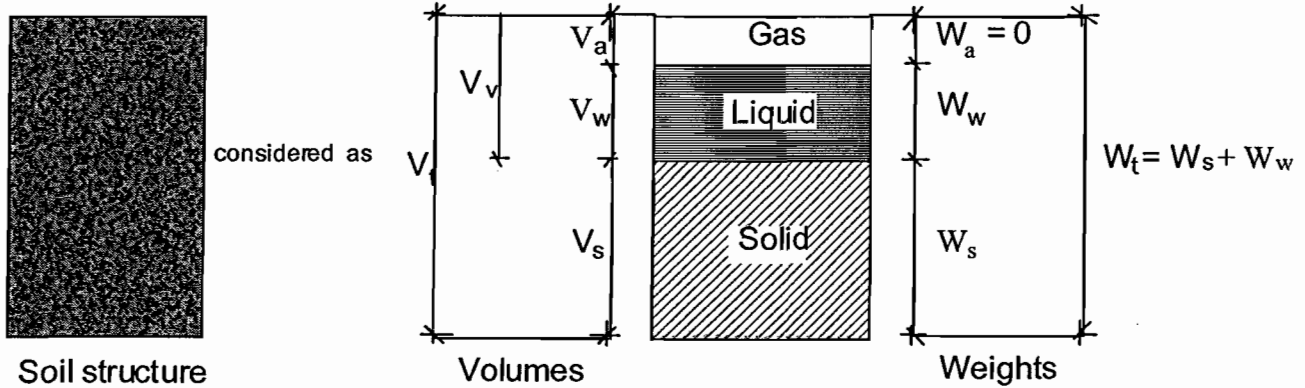
[15%]

**END OF PAPER**

**Engineering Tripos Part IIA****3D1 & 3D2  
Soil Mechanics Data Book  
Data Book 2005/2006**

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## General definitions



Specific gravity of solid

$$G_s$$

Voids ratio

$$e = V_v / V_s$$

Specific volume

$$v = V_t / V_s = 1 + e$$

Porosity

$$n = V_v / V_t = e / (1 + e)$$

Water content

$$w = (W_w / W_s)$$

Degree of saturation

$$S_r = V_w / V_v = (w G_s / e)$$

Unit weight of water

$$\gamma_w = 9.81 \text{ kN/m}^3$$

Unit weight of soil

$$\gamma = W_t / V_t = \left( \frac{G_s + S_r e}{1 + e} \right) \gamma_w$$

Buoyant saturated unit weight

$$\gamma' = \gamma - \gamma_w = \left( \frac{G_s - 1}{1 + e} \right) \gamma_w$$

Unit weight of dry solids

$$\gamma_d = W_s / V_t = \left( \frac{G_s}{1 + e} \right) \gamma_w$$

Air volume ratio

$$A = V_a / V_t = \left( \frac{e(1 - S_r)}{1 + e} \right)$$



**Soil classification (BS1377)**Liquid limit  $w_L$ Plastic Limit  $w_P$ Plasticity Index  $I_P = w_L - w_P$ Liquidity Index  $I_L = \frac{w - w_P}{w_L - w_P}$ Activity =  $\frac{\text{Plasticity Index}}{\text{Percentage of particles finer than } 2 \mu\text{m}}$ Sensitivity =  $\frac{\text{Unconfined compressive strength of an undisturbed specimen}}{\text{Unconfined compressive strength of a remoulded specimen}}$  (at the same water content)*Classification of particle sizes:-*

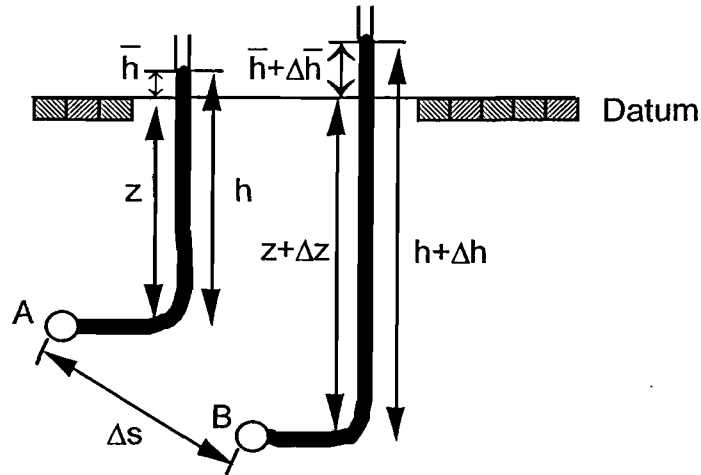
Boulders	larger than			200 mm
Cobbles	between	200 mm	and	60 mm
Gravel	between	60 mm	and	2 mm
Sand	between	2 mm	and	0.06 mm
Silt	between	0.06 mm	and	0.002 mm
Clay	smaller than	0.002 mm (two microns)		

D equivalent diameter of soil particle

 $D_{10}$ ,  $D_{60}$  etc. particle size such that 10% (or 60%) etc.) by weight of a soil sample is composed of finer grains. $C_U$  uniformity coefficient  $D_{60} / D_{10}$

## Seepage

Flow potential:  
(piezometric level)



Total gauge pore water pressure at A:  $u = \gamma_w h = \gamma_w (\bar{h} + z)$

B:  $u + \Delta u = \gamma_w (h + \Delta h) = \gamma_w (\bar{h} + z + \Delta \bar{h} + \Delta z)$

Excess pore water pressure at A:  $\bar{u} = \gamma_w \bar{h}$

B:  $\bar{u} + \Delta \bar{u} = \gamma_w (\bar{h} + \Delta \bar{h})$

Hydraulic gradient A  $\rightarrow$  B  $i = -\frac{\Delta \bar{h}}{\Delta s}$

Hydraulic gradient (3D)  $i = -\nabla \bar{h}$

Darcy's law  $V = ki$

$V$  = superficial seepage velocity

$k$  = coefficient of permeability

Typical permeabilities:

$D_{10} > 10 \text{ mm}$	:	non-laminar flow
$10 \text{ mm} > D_{10} > 1 \mu\text{m}$	:	$k \cong 0.01 (D_{10} \text{ in mm})^2 \text{ m/s}$
clays	:	$k \cong 10^{-9} \text{ to } 10^{-11} \text{ m/s}$

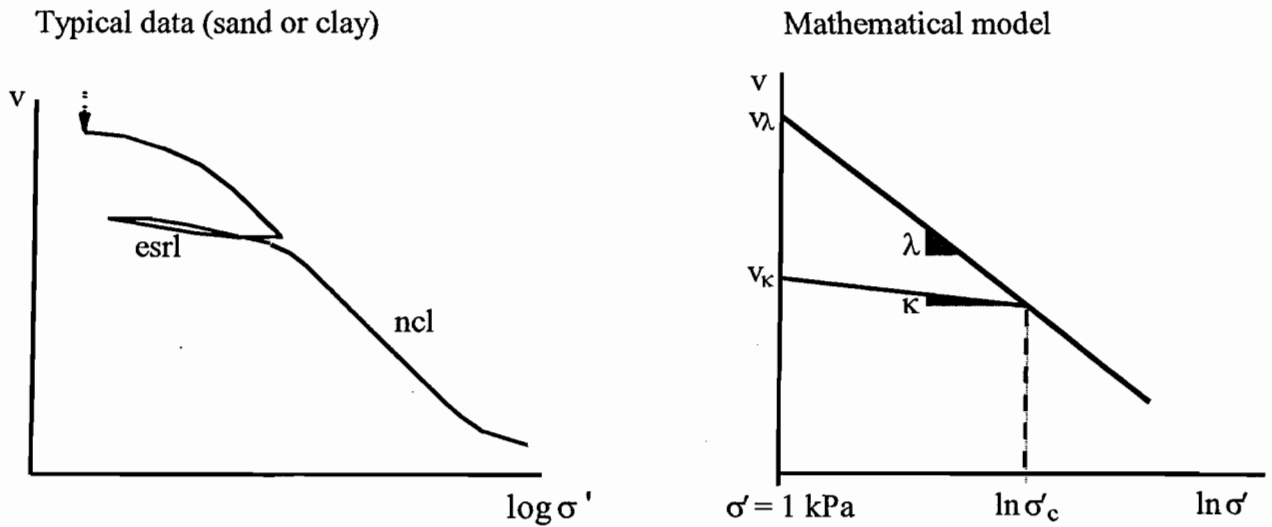
Saturated capillary zone

$h_c = \frac{4T}{\gamma_w d}$  : capillary rise in tube diameter  $d$ , for surface tension  $T$

$h_c \approx \frac{3 \times 10^{-5}}{D_{10}} \text{ m}$  : for water at  $10^\circ\text{C}$ ; note air entry suction is  $u_c = -\gamma_w h_c$

## One-Dimensional Compression

### • Fitting data



Plastic compression stress  $\sigma'_c$  is taken as the larger of the initial aggregate crushing stress and the historic maximum effective vertical stress. Clay muds are taken to begin with  $\sigma'_c \approx 1$  kPa.

Plastic compression (normal compression line, ncl):  $v = v_\lambda - \lambda \ln \sigma'$  for  $\sigma' = \sigma'_c$

Elastic swelling and recompression line (esrl):  
 $v = v_c + \kappa (\ln \sigma'_c - \ln \sigma'_v)$   
 $= v_\kappa - \kappa \ln \sigma'_v$  for  $\sigma' < \sigma'_c$

Equivalent parameters for  $\log_{10}$  stress scale:

Terzaghi's compression index  $C_c = \lambda \log_{10} e$

Terzaghi's swelling index  $C_s = \kappa \log_{10} e$

### • Deriving confined soil stiffnesses

Secant 1D compression modulus  $E_o = (\Delta \sigma' / \Delta \epsilon)_o$

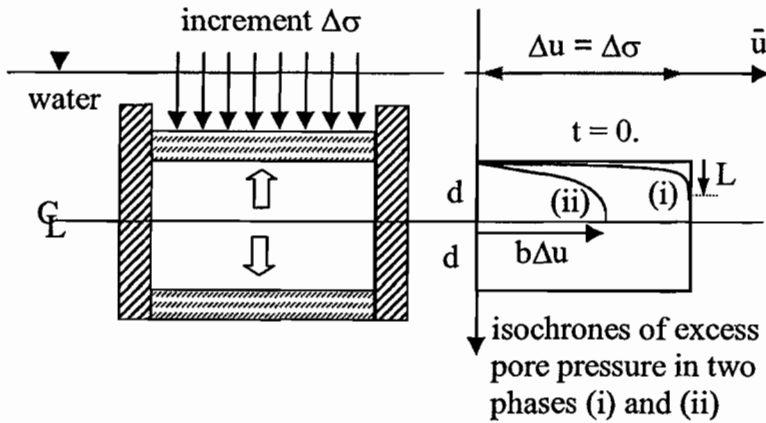
Tangent 1D plastic compression modulus  $E_o = v \sigma' / \lambda$

Tangent 1D elastic compression modulus  $E_o = v \sigma' / \kappa$

### One-Dimensional Consolidation

Settlement	$\rho$	$= \int m_v(\Delta u - \bar{u}) dz$	$= \int (\Delta u - \bar{u}) / E_o dz$
Coefficient of consolidation	$c_v$	$= \frac{k}{m_v \gamma_w}$	$= \frac{kE_o}{\gamma_w}$
Dimensionless time factor	$T_v$	$= \frac{c_v t}{d^2}$	
Relative settlement	$R_v$	$= \frac{\rho}{\rho_{ult}}$	

• Solutions for initially rectangular distribution of excess pore pressure



Approximate solution by parabolic isochrones:

Phase (i)  $L^2 = 12 c_v t$   
 $R_v = \sqrt{\frac{4T_v}{3}}$  for  $T_v < 1/12$

Phase (ii)  $b = \exp(1/4 - 3T_v)$   
 $R_v = [1 - 2/3 \exp(1/4 - 3T_v)]$  for  $T_v > 1/12$

Solution by Fourier Series:

$T_v$	0	0.01	0.02	0.04	0.08	0.15	0.20	0.30	0.40	0.50	0.60	0.80	1.00
$R_v$	0	0.12	0.17	0.23	0.32	0.45	0.51	0.62	0.70	0.77	0.82	0.89	0.94

## Stress and strain components

- **Principle of effective stress (saturated soil)**

$$\text{total stress } \sigma = \text{effective stress } \sigma' + \text{pore water pressure } u$$

- **Principal components of stress and strain**

sign convention	compression positive
total stress	$\sigma_1, \sigma_2, \sigma_3$
effective stress	$\sigma'_1, \sigma'_2, \sigma'_3$
strain	$\varepsilon_1, \varepsilon_2, \varepsilon_3$

- **Simple Shear Apparatus (SSA)** ( $\varepsilon_2 = 0$ ; other principal directions unknown)

The only stresses that are readily available are the shear stress  $\tau$  and normal stress  $\sigma$  applied to the top platen. The pore pressure  $u$  can be controlled and measured, so the normal effective stress  $\sigma'$  can be found. Drainage can be permitted or prevented. The shear strain  $\gamma$  and normal strain  $\varepsilon$  are measured with respect to the top platen, which is a plane of zero extension. Zero extension planes are often identified with slip surfaces.

$$\text{work increment per unit volume} \quad \delta W = \tau \delta\gamma + \sigma' \delta\varepsilon$$

- **Biaxial Apparatus - Plane Strain (BA-PS)** ( $\varepsilon_2 = 0$ ; rectangular edges along principal axes)

Intermediate principal effective stress  $\sigma'_2$ , in zero strain direction, is frequently unknown so that all conditions are related to components in the 1-3 plane.

mean total stress	$s = (\sigma_1 + \sigma_3)/2$
mean effective stress	$s' = (\sigma'_1 + \sigma'_3)/2 = s - u$
shear stress	$t = (\sigma'_1 - \sigma'_3)/2 = (\sigma_1 - \sigma_3)/2$
volumetric strain	$\varepsilon_v = \varepsilon_1 + \varepsilon_3$
shear strain	$\varepsilon_\gamma = \varepsilon_1 - \varepsilon_3$
work increment per unit volume	$\delta W = \sigma'_1 \delta\varepsilon_1 + \sigma'_3 \delta\varepsilon_3$
	$\delta W = s' \delta\varepsilon_v + t \delta\varepsilon_\gamma$

providing that principal axes of strain increment and of stress coincide.

• **Triaxial Apparatus – Axial Symmetry (TA-AS)** (cylindrical element with radial symmetry)

total axial stress	$\sigma_a = \sigma'_a + u$
total radial stress	$\sigma_r = \sigma'_r + u$
total mean normal stress	$p = (\sigma_a + 2\sigma_r)/3$
effective mean normal stress	$p' = (\sigma'_a + 2\sigma'_r)/3 = p - u$
deviatoric stress	$q = \sigma'_a - \sigma'_r = \sigma_a - \sigma_r$
stress ratio	$\eta = q/p'$
axial strain	$\epsilon_a$
radial strain	$\epsilon_r$
volumetric strain	$\epsilon_v = \epsilon_a + 2\epsilon_r$
triaxial shear strain	$\epsilon_s = \frac{2}{3}(\epsilon_a - \epsilon_r)$
work increment per unit volume	$\delta W = \sigma'_a \delta \epsilon_a + 2\sigma'_r \delta \epsilon_r$
	$\delta W = p' \delta \epsilon_v + q \delta \epsilon_s$

Types of triaxial test include:

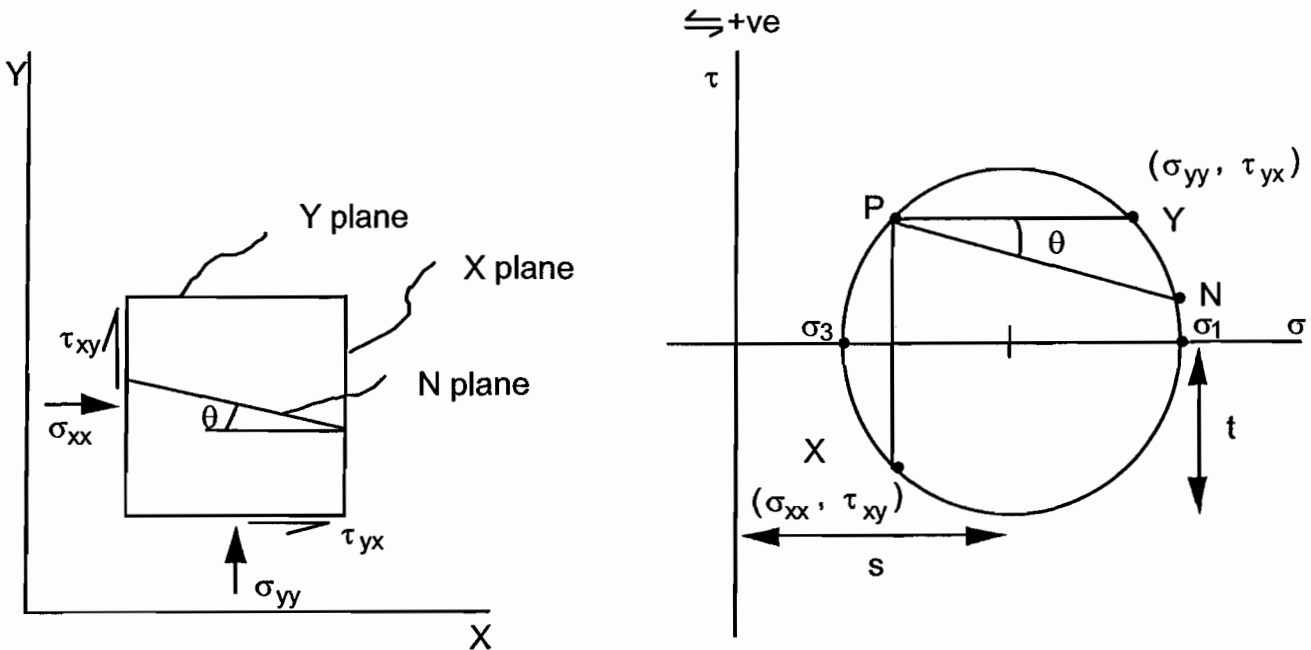
*isotropic compression* in which  $p'$  increases at zero  $q$

*triaxial compression* in which  $q$  increases *either* by increasing  $\sigma_a$  *or* by reducing  $\sigma_r$

*triaxial extension* in which  $q$  reduces *either* by reducing  $\sigma_a$  *or* by increasing  $\sigma_r$

• **Mohr's circle of stress (1–3 plane)**

Sign of convention: compression, and counter-clockwise shear, positive



*Poles of planes P*: the components of stress on the N plane are given by the intersection N of the Mohr circle with the line PN through P parallel to the plane.

## Elastic stiffness relations

These relations apply to tangent stiffnesses of over-consolidated soil, with a state point on some swelling and recompression line ( $\kappa$ -line), and remote from gross plastic yielding.

One-dimensional compression (axial stress and strain increments  $d\sigma'$ ,  $d\varepsilon$ )

$$\text{compressibility} \quad m_v = \frac{d\varepsilon}{d\sigma'}$$

$$\text{constrained modulus} \quad E_o = \frac{1}{m_v}$$

Physically fundamental parameters

$$\text{shear modulus} \quad G' = \frac{dt}{d\varepsilon_\gamma}$$

$$\text{bulk modulus} \quad K' = \frac{dp'}{d\varepsilon_v}$$

Parameters which can be used for constant-volume deformations

$$\text{undrained shear modulus} \quad G_u = G'$$

$$\text{undrained bulk modulus} \quad K_u = \infty \quad (\text{neglecting compressibility of water})$$

Alternative convenient parameters

$$\text{Young's moduli} \quad E' \text{ (effective), } E_u \text{ (undrained)}$$

$$\text{Poisson's ratios} \quad \nu' \text{ (effective), } \nu_u = 0.5 \text{ (undrained)}$$

Typical value of Poisson's ratio for small changes of stress:  $\nu' = 0.2$

$$\text{Relationships:} \quad G = \frac{E}{2(1+\nu)}$$

$$K = \frac{E}{3(1-2\nu)}$$

$$E_o = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

## Cam Clay

• Interchangeable parameters for stress combinations at yield, and plastic strain increments

System	Effective normal stress	Plastic normal strain	Effective shear stress	Plastic shear strain	Critical stress ratio	Plastic normal stress	Critical normal stress
General	$\sigma^*$	$\varepsilon^*$	$\tau^*$	$\gamma^*$	$\mu^*_{crit}$	$\sigma^*_c$	$\sigma^*_{crit}$
SSA	$\sigma'$	$\varepsilon$	$\tau$	$\gamma$	$\tan \phi_{crit}$	$\sigma'_c$	$\sigma'_{crit}$
BA-PS	$s'$	$\varepsilon_v$	$t$	$\varepsilon_\gamma$	$\sin \phi_{crit}$	$s'_c$	$s'_{crit}$
TA-AS	$p'$	$\varepsilon_v$	$q$	$\varepsilon_s$	$M$	$p'_c$	$p'_{crit}$

• General equations of plastic work

Plastic work and dissipation

$$\sigma^* \delta\varepsilon^* + \tau^* \delta\gamma^* = \mu^*_{crit} \sigma^* \delta\gamma^*$$

Plastic flow rule – normality

$$\frac{d\tau^*}{d\sigma^*} \cdot \frac{d\gamma^*}{d\varepsilon^*} = -1$$

• General yield surface

$$\frac{\tau^*}{\sigma^*} = \mu^* = \mu^*_{crit} \cdot \ln \left[ \frac{\sigma_c^*}{\sigma^*} \right]$$

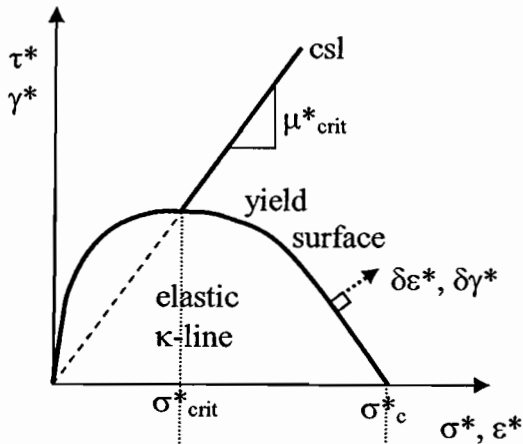
• Parameter values which fit soil data

	London Clay	Weald Clay	Kaolin	Dog's Bay Sand	Ham River Sand
$\lambda^*$	0.161	0.093	0.26	0.334	0.163
$\kappa^*$	0.062	0.035	0.05	0.009	0.015
$\Gamma^*$ at 1 kPa	2.759	2.060	3.767	4.360	3.026
$\sigma^*_{c, virgin}$ kPa	1	1	1	Loose 500 Dense 1500	Loose 2500 Dense 15000
$\phi_{crit}$	23°	24°	26°	39°	32°
$M_{comp}$	0.89	0.95	1.02	1.60	1.29
$M_{extn}$	0.69	0.72	0.76	1.04	0.90
$w_L$	0.78	0.43	0.74	-----	-----
$w_P$	0.26	0.18	0.42	-----	-----
$G_s$	2.75	2.75	2.61	2.75	2.65

Note: 1) parameters  $\lambda^*$ ,  $\kappa^*$ ,  $\Gamma^*$ ,  $\sigma^*_{c, virgin}$  should depend to a small extent on the deformation mode, e.g. SSA, BA-PS, TA-AS, etc. This may be neglected unless further information is given.  
 2) Sand which is loose, or loaded cyclically, compacts more than Cam Clay allows.



• The yield surface in  $(\sigma^*, \tau^*, v)$  space



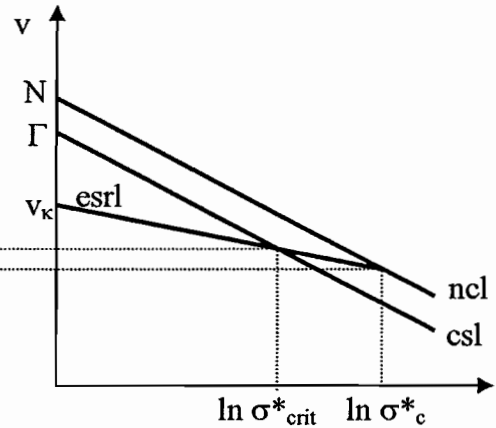
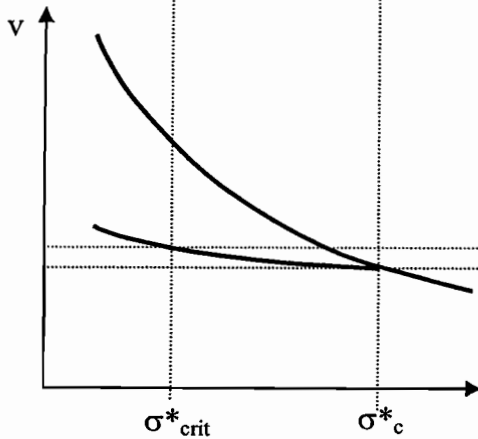
ncl: normal compression line

$$v = N - \lambda \ln \sigma^*$$

csl: critical state line

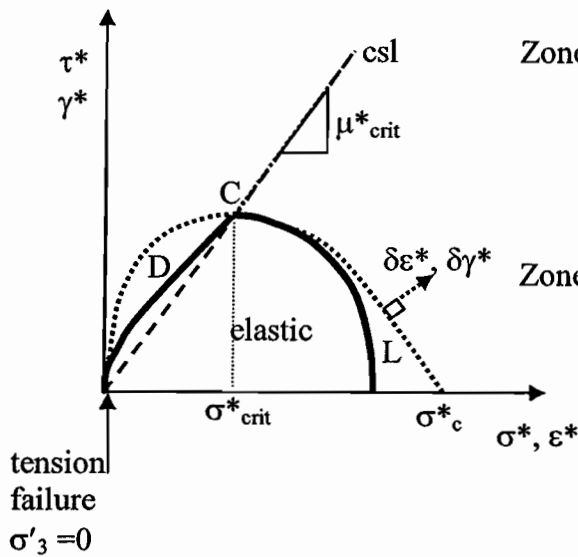
$$v = \Gamma - \lambda \ln \sigma^*$$

where  $N = \Gamma + \lambda - \kappa$



• Regions of limiting soil behaviour

Variation of Cam Clay yield surface

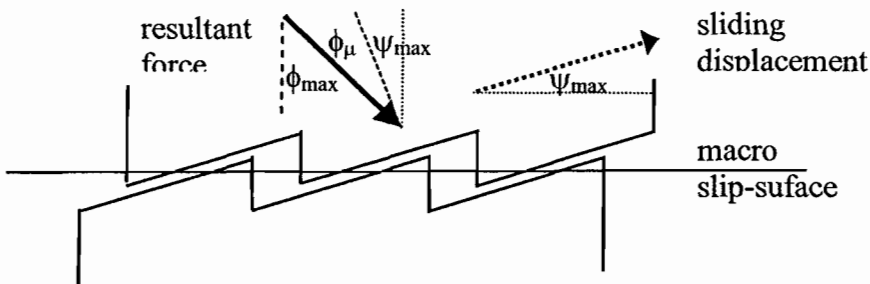


Zone D: denser than critical, "dry",  
dilation or negative excess pore pressures,  
Hvorslev strength envelope,  
friction-dilatancy theory,  
unstable shear rupture, progressive failure

Zone L: looser than critical, "wet",  
compaction or positive excess pore pressures,  
Modified Cam Clay yield surface,  
stable strain-hardening continuum

## Strength of soil: friction and dilation

### • Friction and dilatancy: the saw-blade model of direct shear

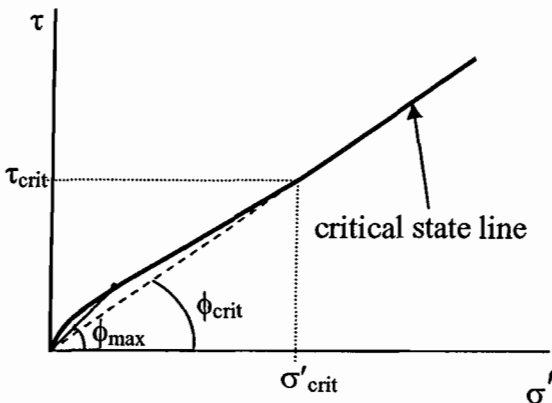


Intergranular angle of friction at sliding contacts  $\phi_{\mu}$

Angle of dilation  $\psi_{max}$

Angle of internal friction  $\phi_{max} = \phi_{\mu} + \psi_{max}$

### • Friction and dilatancy: secant and tangent strength parameters

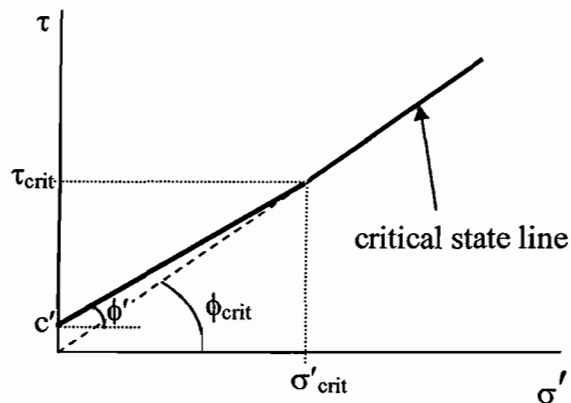


Secant angle of internal friction

$$\begin{aligned}\tau &= \sigma' \tan \phi_{max} \\ \phi_{max} &= \phi_{crit} + \Delta\phi \\ \Delta\phi &= f(\sigma'_{crit}/\sigma')\end{aligned}$$

typical envelope fitting data:

$$\begin{aligned}\text{power curve} \\ (\tau/\tau_{crit}) &= (\sigma'/\sigma'_{crit})^{\alpha} \\ \text{with } \alpha &\approx 0.85\end{aligned}$$



Tangent angle of shearing envelope

$$\begin{aligned}\tau &= c' + \sigma' \tan \phi' \\ c' &= f(\sigma'_{crit})\end{aligned}$$

typical envelope:

$$\begin{aligned}\text{straight line} \\ \tan \phi' &= 0.85 \tan \phi_{crit} \\ c' &= 0.15 \tau_{crit}\end{aligned}$$

• **Friction and dilation: data of sands**

The inter-granular friction angle of quartz grains,  $\phi_\mu \approx 26^\circ$ . Turbulent shearing at a critical state causes  $\phi_{crit}$  to exceed this. The critical state angle of internal friction  $\phi_{crit}$  is a function of the uniformity of particle sizes, their shape, and mineralogy, and is developed at large shear strains irrespective of initial conditions. Typical values of  $\phi_{crit} (\pm 2^\circ)$  are:

well-graded, angular quartz or feldspar sands	40°
uniform sub-angular quartz sand	36°
uniform rounded quartz sand	32°

Relative density  $I_D = \frac{(e_{max} - e)}{(e_{max} - e_{min})}$  where:

$e_{max}$  is the maximum void ratio achievable in quick-tilt test  
 $e_{min}$  is the minimum void ratio achievable by vibratory compaction

Relative crushability  $I_C = \ln(\sigma_c / p')$  where:

$\sigma_c$  is the aggregate crushing stress, taken to be a material constant, typical values being:  
 80 000 kPa for quartz silt, 20 000 kPa for quartz sand, 5 000 kPa for carbonate sand.

$p'$  is the mean effective stress at failure which may be taken as approximately equal to the effective stress  $\sigma'$  normal to a shear plane.

Dilatancy contribution to the peak angle of internal friction is  $\Delta\phi = (\phi_{max} - \phi_{crit}) = f(I_R)$

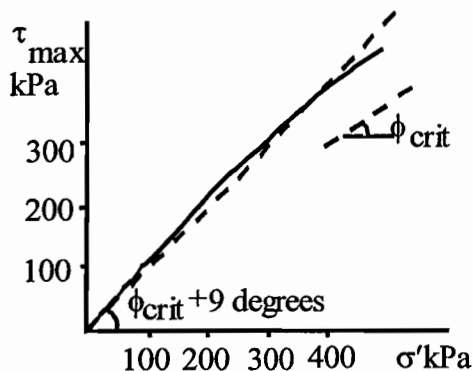
Relative dilatancy index  $I_R = I_D I_C - 1$  where:

$I_R < 0$  indicates compaction, so that  $I_D$  increases and  $I_R \rightarrow 0$  ultimately at a critical state  
 $I_R > 4$  to be limited to  $I_R = 4$  unless corroborative dilatant strength data is available

The following empirical correlations are then available

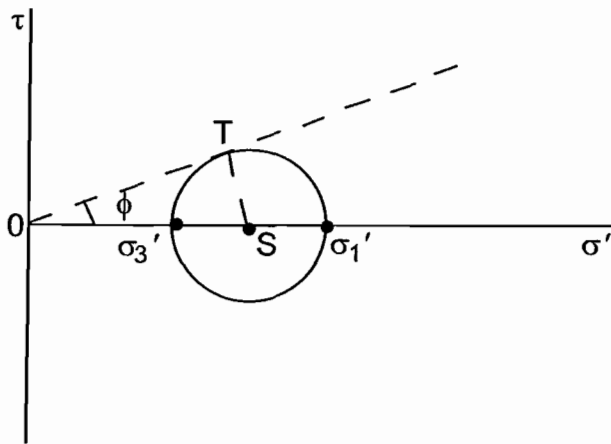
plane strain conditions	$(\phi_{max} - \phi_{crit})$	=	0.8 $\psi_{max}$	=	5 $I_R$ degrees
triaxial strain conditions	$(\phi_{max} - \phi_{crit})$	=	3 $I_R$ degrees		
all conditions	$(-\delta\varepsilon_v / \delta\varepsilon_1)_{max}$	=	0.3 $I_R$		

The resulting peak strength envelope for triaxial tests on a quartz sand at an initial relative density  $I_D = 1$  is shown below for the limited stress range 10 - 400 kPa:



$\phi_{max} > \phi_{crit} + 9^\circ$  for  $I_D = 1, \sigma' = 400$  kPa

• Mobilised (secant) angle of shearing  $\phi$  in the 1 – 3 plane



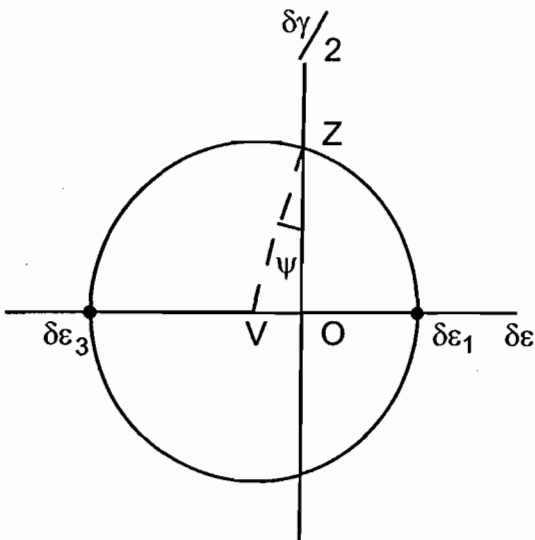
$$\begin{aligned} \sin \phi &= TS/OS \\ &= \frac{(\sigma_1' - \sigma_3')/2}{(\sigma_1' + \sigma_3')/2} \\ \left[ \frac{\sigma_1'}{\sigma_3'} \right] &= \frac{(1 + \sin \phi)}{(1 - \sin \phi)} \end{aligned}$$

Angle of shearing resistance:

at peak strength  $\phi_{\max}$  at  $\left[ \frac{\sigma_1'}{\sigma_3'} \right]_{\max}$

at critical state  $\phi_{\text{crit}}$  after large shear strains

• Mobilised angle of dilation in plane strain  $\psi$  in the 1 – 3 plane



$$\begin{aligned} \sin \psi &= VO/VZ \\ &= - \frac{(\delta \epsilon_1 + \delta \epsilon_3)/2}{(\delta \epsilon_1 - \delta \epsilon_3)/2} \\ &= - \frac{\delta \epsilon_v}{\delta \epsilon_\gamma} \end{aligned}$$

$$\left[ \frac{\delta \epsilon_1}{\delta \epsilon_3} \right] = - \frac{(1 - \sin \psi)}{(1 + \sin \psi)}$$

at peak strength  $\psi = \psi_{\max}$  at  $\left[ \frac{\sigma_1'}{\sigma_3'} \right]_{\max}$

at critical state  $\psi = 0$  since volume is constant

**Plasticity: Cohesive material  $\tau_{max} = c_u$  (or  $s_u$ )**

• **Limiting stresses**

Tresca  $|\sigma_1 - \sigma_3| = q_u = 2c_u$

von Mises  $(\sigma_1 - p)^2 + (\sigma_2 - p)^2 + (\sigma_3 - p)^2 = \frac{2}{3} q_u^2 = 2c_u^2$

where  $q_u$  is the undrained triaxial compression strength, and  $c_u$  is the undrained plane shear strength.

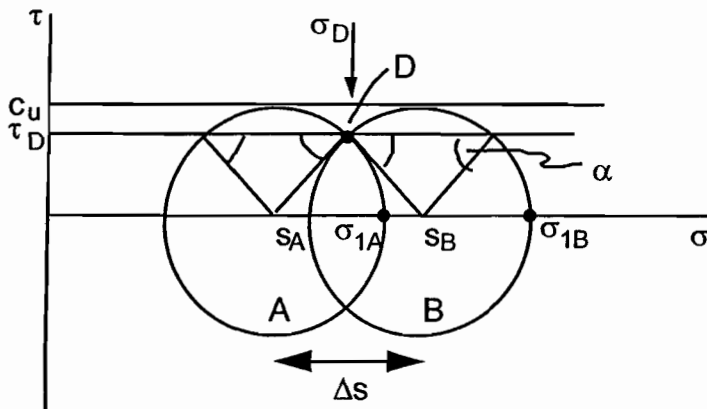
Dissipation per unit volume in plane strain deformation following either Tresca or von Mises,

$$\delta D = c_u \delta \epsilon_\gamma$$

For a relative displacement  $x$  across a slip surface of area  $A$  mobilising shear strength  $c_u$ , this becomes

$$D = Ac_u x$$

• **Stress conditions across a discontinuity**



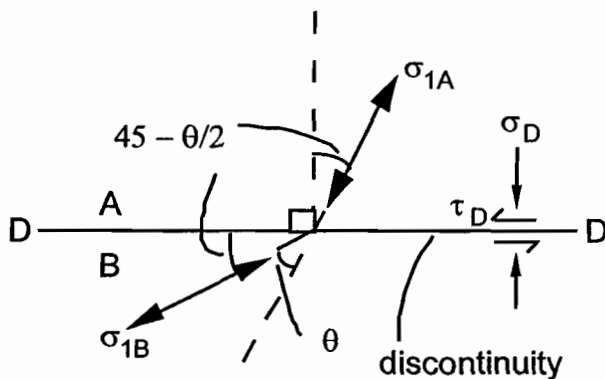
Rotation of major principal stress  $\theta$

$$s_B - s_A = \Delta s = 2c_u \sin \theta$$

$$\sigma_{1B} - \sigma_{1A} = 2c_u \sin \theta$$

In limit with  $\theta \rightarrow 0$

$$ds = 2c_u d\theta$$



Useful example:

$$\theta = 30^\circ$$

$$\sigma_{1B} - \sigma_{1A} = c_u$$

$$\tau_D / c_u = 0.87$$

$\sigma_{1A}$  = major principal stress in zone A

$\sigma_{1B}$  = major principal stress in zone B

**Plasticity: Frictional material**  $(\tau/\sigma')_{\max} = \tan \phi'$

• **Limiting stresses**

$$\sin \phi = (\sigma'_{1f} - \sigma'_{3f}) / (\sigma'_{1f} + \sigma'_{3f}) = (\sigma_{1f} - \sigma_{3f}) / (\sigma_{1f} + \sigma_{3f} - 2u_s)$$

where  $\sigma'_{1f}$  and  $\sigma'_{3f}$  are the major and minor principal effective stresses at failure,  $\sigma_{1f}$  and  $\sigma_{3f}$  are the major and minor principle total stresses at failure, and  $u_s$  is the steady state pore pressure.

Active pressure:

$$\sigma'_v > \sigma'_h$$

$$\sigma'_1 = \sigma'_v \text{ (assuming principal stresses are horizontal and vertical)}$$

$$\sigma'_3 = \sigma'_h$$

$$K_a = (1 - \sin \phi) / (1 + \sin \phi)$$

Passive pressure:

$$\sigma'_h > \sigma'_v$$

$$\sigma'_1 = \sigma'_h \text{ (assuming principal stresses are horizontal and vertical)}$$

$$\sigma'_3 = \sigma'_v$$

$$K_p = (1 + \sin \phi) / (1 - \sin \phi) = 1 / K_a$$

• **Stress conditions across a discontinuity**

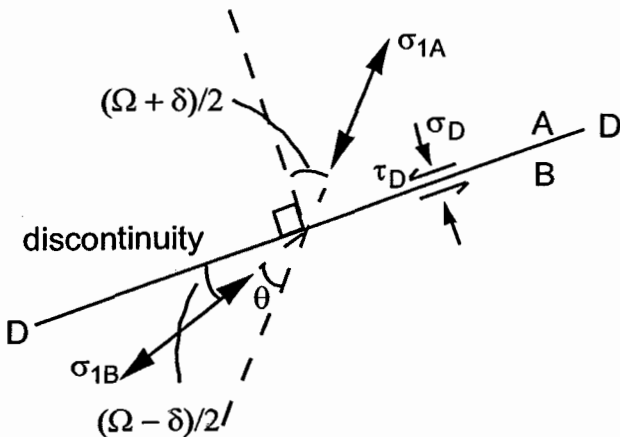
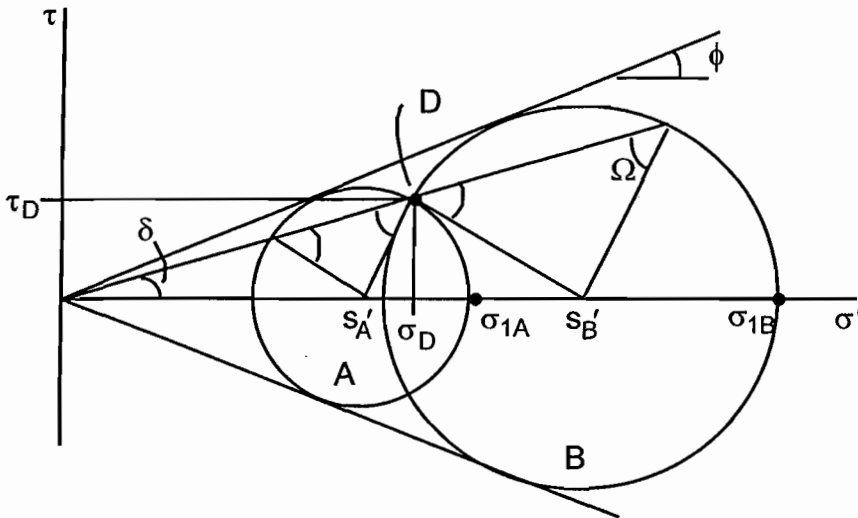
Rotation of major principal stress

$$\theta = \pi/2 - \Omega$$

$\sigma_{1A}$  = major principal stress in zone A

$\sigma_{1B}$  = major principal stress in zone B

$$\tan \delta = \tau_D / \sigma'_D$$



$$\sin \Omega = \sin \delta / \sin \phi$$

$$s'_B / s'_A = \sin(\Omega + \delta) / \sin(\Omega - \delta)$$

In limit,  $d\theta \rightarrow 0$  and  $\delta \rightarrow \phi$

$$ds' = 2s' \cdot d\theta \tan \phi$$

Integration gives  $s'_B / s'_A = \exp(2\theta \tan \phi)$

## Empirical earth pressure coefficients following one-dimensional strain

Coefficient of earth pressure in 1D plastic compression (normal compression)

$$K_{o,nc} = 1 - \sin \phi_{crit}$$

Coefficient of earth pressure during a 1D unloading-reloading cycle (overconsolidated soil)

$$K_o = K_{o,nc} \left[ 1 + \frac{(n-1)(n_{max}^\alpha - 1)}{(n_{max} - 1)} \right]$$

where  $n$  is current overconsolidation ratio (OCR) defined as  $\sigma'_{v,max} / \sigma'_v$

$n_{max}$  is maximum historic OCR defined as  $\sigma'_{v,max} / \sigma'_{v,min}$

$\alpha$  is to be taken as  $1.2 \sin \phi_{crit}$

## Cylindrical cavity expansion

Expansion  $\delta A = A - A_o$  caused by increase of pressure  $\delta \sigma_c = \sigma_c - \sigma_o$

At radius  $r$ : small displacement  $\rho = \frac{\delta A}{2\pi r}$

small shear strain  $\gamma = \frac{2\rho}{r}$

Radial equilibrium:  $r \frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta = 0$

Elastic expansion (small strains)  $\delta \sigma_c = G \frac{\delta A}{A}$

Undrained plastic-elastic expansion  $\delta \sigma_c = c_u \left[ 1 + \ln \frac{G}{c_u} + \ln \frac{\delta A}{A} \right]$

## Shallow foundation design

### *Tresca soil, with undrained strength $s_u$*

#### Vertical loading

The vertical bearing capacity,  $q_f$ , of a shallow foundation for undrained loading (Tresca soil) is:

$$\frac{V_{ult}}{A} = q_f = s_c d_c N_c s_u + \gamma h$$

$V_{ult}$  and  $A$  are the ultimate vertical load and the foundation area, respectively.  $h$  is the embedment of the foundation base and  $\gamma$  (or  $\gamma'$ ) is the appropriate density of the overburden.

The exact bearing capacity factor  $N_c$  for a plane strain surface foundation (zero embedment) on uniform soil is:

$$N_c = 2 + \pi \quad (\text{Prandtl, 1921})$$

#### *Shape correction factor:*

For a rectangular footing of length  $L$  and breadth  $B$  (Eurocode 7):

$$s_c = 1 + 0.2 B / L$$

The exact solution for a rough circular foundation ( $D = B = L$ ) is  $q_f = 6.05s_u$ , hence  $s_c = 1.18 \sim 1.2$ .

#### *Embedment correction factor:*

A fit to Skempton's (1951) embedment correction factors, for an embedment of  $h$ , is:

$$d_c = 1 + 0.33 \tan^{-1} (h/B) \quad (\text{or } h/D \text{ for a circular foundation})$$

#### **Combined V-H loading**

A curve fit to Green's lower bound plasticity solution for V-H loading is:

$$\text{If } V/V_{ult} > 0.5: \quad \frac{V}{V_{ult}} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{H}{H_{ult}}} \quad \text{or} \quad \frac{H}{H_{ult}} = 1 - \left( 2 \frac{V}{V_{ult}} - 1 \right)^2$$

$$\text{If } V/V_{ult} < 0.5: \quad H = H_{ult} = B s_u$$

#### **Combined V-H-M loading**

With lift-off: combined Green-Meyerhof

$$\text{Without lift-off:} \quad \left( \frac{V}{V_{ult}} \right)^2 + \left[ \frac{M}{M_{ult}} \left( 1 - 0.3 \frac{H}{H_{ult}} \right) \right]^2 + \left| \left( \frac{H}{H_{ult}} \right)^3 \right| - 1 = 0 \quad (\text{Taiebet \& Carter 2000})$$



### Frictional (Coulomb) soil, with friction angle $\phi$

#### Vertical loading

The vertical bearing capacity,  $q_f$ , of a shallow foundation under drained loading (Coulomb soil) is:

$$\frac{V_{ult}}{A} = q_f = s_q N_q \sigma'_{v0} + s_\gamma N_\gamma \frac{\gamma' B}{2}$$

The bearing capacity factors  $N_q$  and  $N_\gamma$  account for the capacity arising from surcharge and self-weight of the foundation soil respectively.  $\sigma'_{v0}$  is the in situ effective stress acting at the level of the foundation base.

For a strip footing on weightless soil, the exact solution for  $N_q$  is:

$$N_q = \tan^2(\pi/4 + \phi/2) e^{(\pi \tan \phi)} \quad (\text{Prandtl 1921})$$

An empirical relationship to estimate  $N_\gamma$  from  $N_q$  is (Eurocode 7):

$$N_\gamma = 2 (N_q - 1) \tan \phi$$

Curve fits to exact solutions for  $N_\gamma = f(\phi)$  are (Davis & Booker 1971):

$$\text{Rough base:} \quad N_\gamma = 0.1054 e^{9.6\phi}$$

$$\text{Smooth base:} \quad N_\gamma = 0.0663 e^{9.3\phi}$$

#### Shape correction factors:

For a rectangular footing of length  $L$  and breadth  $B$  (Eurocode 7):

$$s_q = 1 + (B \sin \phi) / L$$

$$s_\gamma = 1 - 0.3 B / L$$

For circular footings take  $L = B$ .

#### Combined V-H loading

The Green/Sokolovski lower bound solution gives a V-H failure surface.

#### Combined V-H-M loading

With lift-off- drained conditions - use Butterfield & Gottardi (1994) failure surface shown above

$$\left[ \frac{H/V_{ult}}{t_h} \right]^2 + \left[ \frac{M/BV_{ult}}{t_m} \right]^2 + \left[ \frac{2C(M/BV_{ult})(H/V_{ult})}{t_h t_m} \right] = \left[ \frac{V}{V_{ult}} \left( 1 - \frac{V}{V_{ult}} \right) \right]^2$$

$$\text{where } C = \tan \left( \frac{2\rho(t_h - t_m)(t_h + t_m)}{2t_h t_m} \right) \quad (\text{Butterfield \& Gottardi, 1994})$$

Typically,  $t_h \sim 0.5$ ,  $t_m \sim 0.4$  and  $\rho \sim 15^\circ$ . Note that  $t_h$  is the friction coefficient,  $H/V = \tan \phi$ , during sliding.

