ENGINEERING TRIPOS PART IIA

Thursday 10 May 2007 2.30 to 4.00

Module 3D7

FINITE ELEMENT METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Special datasheets (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A planar pin-jointed structure, with dimensions shown in Fig. 1, consists of four straight members. Each member behaves linearly elastically and has axial stiffness EA.

(a) Set up an equilibrium matrix **H** which relates the axial forces in the structure $\mathbf{r} = \{t_I \quad t_{II} \quad t_{III} \quad t_{IV}\}^T$ to the set of external loads $\mathbf{p} = \{p_X \; p_Y\}^T = \{W \quad 0\}^T$ applied to node A. [15%]

(b) Obtain a general solution to the set of equations $\mathbf{H}\,\mathbf{r} = \mathbf{p}$. [40%]

(c) Evaluate **r** if the structure is initially unstressed. [30%]

(d) Show the method that would be used to obtain the changes in the bar forces evaluated in (c) if bar I was heated to give a temperature change ΔT. [15%]

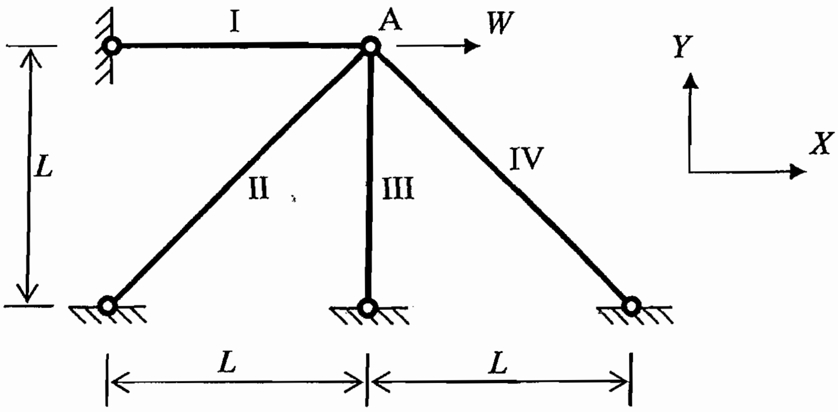

Fig. 1

2 The three node constant strain triangular 2D finite element with co-ordinates shown in Fig. 2 is to be used in a dynamic analysis. The element represents part of a planar sheet of thickness t and density ρ. The equations of motion of this element can be expressed in the form $\mathbf{M}\ddot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{p}$ where **M** is the element mass matrix and **K** is the element stiffness matrix. **d** and **p** are the nodal displacements and corresponding external forces expressed as follows:

$$\mathbf{d} = \{d_{AX} \quad d_{AY} \quad d_{BX} \quad d_{BY} \quad d_{CX} \quad d_{CY}\}^T$$
$$\mathbf{p} = \{p_{AX} \quad p_{AY} \quad p_{BX} \quad p_{BY} \quad p_{CX} \quad p_{CY}\}^T$$

(a) Find a lumped mass matrix **M** for this element. [20%]

(b) Find:

(i) the element shape functions n_A, n_B and n_C; [20%]

(ii) a mass matrix **M** for this element that is consistent with these shape functions. [60%]

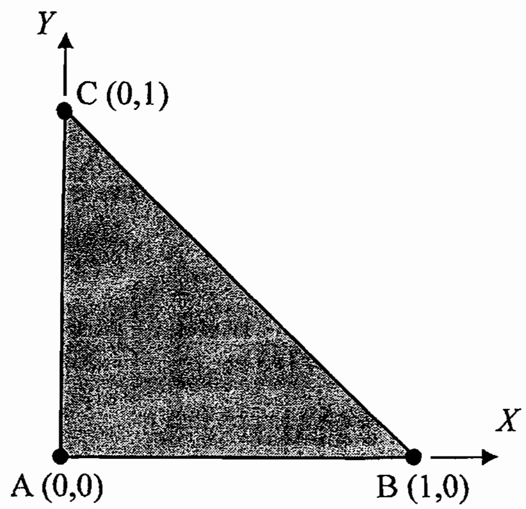

Fig. 2

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3 An object consisting of two solid cylinders stacked together is placed on top of a level flat surface as shown in Fig. 3. The centre lines of the two cylinders coincide.

(a) A uniform temperature of 40°C is applied to the top surface whilst the bottom surface is held at 20°C.

(i) Sketch a finite element model that can be used to evaluate the temperature field inside the object. If the other surfaces of the object are assumed to be insulated define the boundary conditions in the sketch. Sketch the temperature field that you would expect to obtain from the finite element analysis. [30%]

(ii) If the surfaces of the object are now exposed to air at a constant temperature of 25°C (except for the top and bottom surfaces where constant temperatures are applied), what kind of boundary condition should be applied to these surfaces? [10%]

(b) If a uniform loading is applied to one of the surfaces of the bottom cylinder, show two loading cases that can be simulated by an axi-symmetric finite element model. Sketch the cases and define the boundary conditions. Identify the region where a finer finite element mesh should be used. [20%]

(c) A point load P is applied on the top surface as shown in Fig. 3. Sketch a suitable finite element model to analyse this problem. [20%]

(d) Stresses are to be evaluated from the finite element analysis. Should they be evaluated at the Gauss points or nodal points? Why? [20%]

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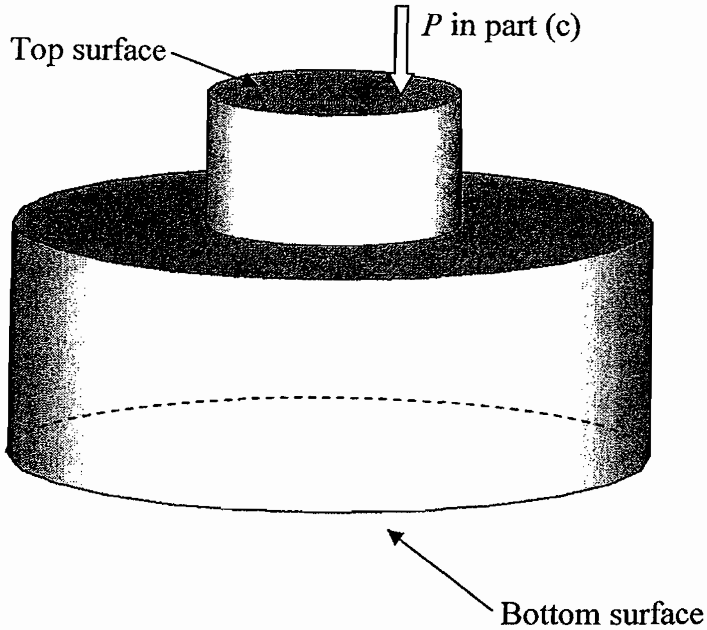

Fig. 3

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4 The finite element mesh shown in Fig. 4 consists of two 3-node constant strain triangular elements I and II. The mesh represents part of a planar sheet of Young's Modulus $E = 100$ kN mm^{-2}, Poisson's ratio $\nu = 0$ and thickness $t = 1$ mm.

(a) Find the shape functions for element I and hence evaluate the strain shape function matrix **B** for element I. [20%]

(b) Evaluate the coefficients of the 6×6 stiffness matrix $\mathbf{K}_\mathrm{I}$ for element I that relates the nodal displacements $\mathbf{d}_\mathrm{I}$ of element I to the corresponding nodal forces $\mathbf{p}_\mathrm{I}$ where

$$\mathbf{d}_\mathrm{I} = \{d_{1X} \quad d_{1Y} \quad d_{2X} \quad d_{2Y} \quad d_{3X} \quad d_{3Y}\}^T$$
$$\mathbf{p}_\mathrm{I} = \{p_{1X} \quad p_{1Y} \quad p_{2X} \quad p_{2Y} \quad p_{3X} \quad p_{3Y}\}^T \; .$$
[40%]

(c) Introduce the boundary conditions for element I shown in Fig. 4 and hence reduce the size of the stiffness matrix $\mathbf{K}_\mathrm{I}$ for element I to 3×3. [10%]

(d) Find the stiffness matrix for the whole (two element) mesh taking account of all boundary conditions shown in Fig. 4. [30%]

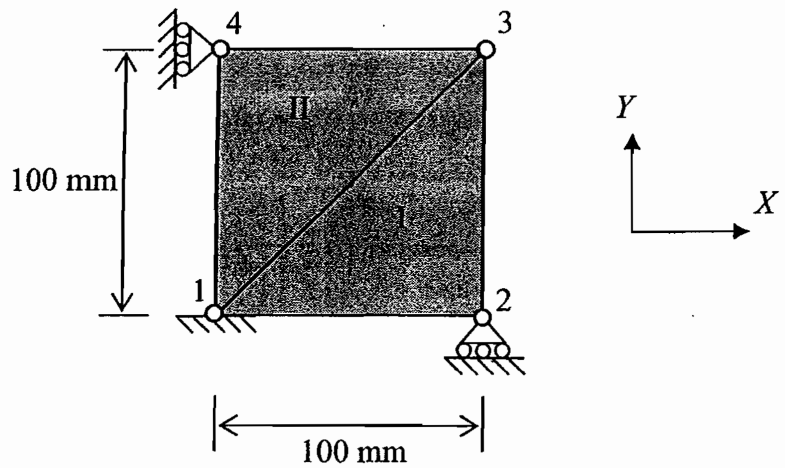

Fig. 4

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Part IIA: Module 3D7 2003-4

Finite Element Methods

Formulae

Force Method

Stress resultants:	solve $\mathbf{Hr} = \mathbf{p}$ and find $\mathbf{r} = \mathbf{r}_0 + \mathbf{Sx}$;'
	then, solve $\mathbf{S}^T\mathbf{FSx} = -\mathbf{S}^T(\mathbf{Fr}_0 + \mathbf{e}_0)$ for $\mathbf{x}$.
Displacements:	solve $\mathbf{H}^T\mathbf{d} = \mathbf{e}$, where $\mathbf{e} = \mathbf{Fr} + \mathbf{e}_0$.

Displacement Method

Displacements:	solve $\mathbf{Kd} = \mathbf{p}$.
Stress resultants:	for element i, solve $\mathbf{F}_i\mathbf{r}_i = \mathbf{e}_i$, where $\mathbf{e}_i = (\mathbf{H}_i')^T\mathbf{d}_i'$.

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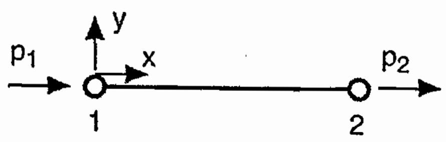
February 2004

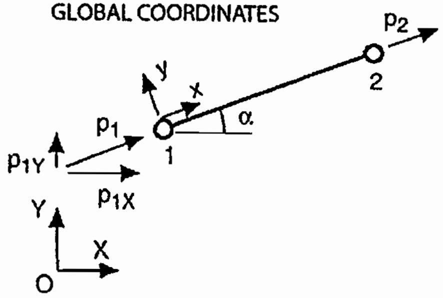
PIN-JOINTED BAR in LOCAL COORDINATES



Static variables	Kinematic variables	Equilibrium	Elasticity	Stiffness $\mathbf{K}_i = \mathbf{H}_i\mathbf{F}_i^{-1}\mathbf{H}_i^T$
$\mathbf{r}_i = [t]$ t = axial force $\mathbf{p}_i = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$	$\mathbf{e}_i = [e]$ e = extension $\mathbf{d}_i = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$	$\mathbf{H}_i = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$	$\mathbf{F}_i = [a]$	$\mathbf{K}_i = \begin{bmatrix} 1/a & -1/a \\ -1/a & 1/a \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness	$\mathbf{H}_i\mathbf{r}_i = \mathbf{p}_i$ $\mathbf{H}_i^T\mathbf{d}_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}_i\mathbf{d}_i = \mathbf{p}_i$			

$a = L/AE$, AE = axial stiffness

PIN-JOINTED BAR in GLOBAL COORDINATES



Static variables	Kinematic variables	Coordinate transformation	Equilibrium	Stiffness
$\mathbf{r}_i$ $\mathbf{p}'_i = \begin{bmatrix} p_{1X} \\ p_{1Y} \\ p_{2X} \\ p_{2Y} \end{bmatrix}$	$\mathbf{e}_i$ $\mathbf{d}'_i = \begin{bmatrix} d_{1X} \\ d_{1Y} \\ d_{2X} \\ d_{2Y} \end{bmatrix}$	$\mathbf{T}_i = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix}$ $\mathbf{R} = \begin{bmatrix} u \\ v \end{bmatrix}$	$\mathbf{H}'_i = \begin{bmatrix} -u \\ -v \\ u \\ v \end{bmatrix}$	$\mathbf{K}'_i = \frac{1}{a}\begin{bmatrix} u^2 & uv & -u^2 & -uv \\ & v^2 & -uv & -v^2 \\ & & u^2 & uv \\ \text{symm.} & & & v^2 \end{bmatrix}$
Equilibrium Compatibility Constitutive Stiffness Transformations	$\mathbf{H}'_i\mathbf{r}_i = \mathbf{p}'_i$ $\mathbf{H}_i'^T\mathbf{d}'_i = \mathbf{e}_i$ $\mathbf{F}_i\mathbf{r}_i + \mathbf{e}_{i0} = \mathbf{e}_i$ $\mathbf{K}'_i\mathbf{d}'_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{p}_i = \mathbf{p}'_i$ $\mathbf{T}_i\mathbf{d}_i = \mathbf{d}'_i$ $\mathbf{T}_i\mathbf{H}_i = \mathbf{H}'_i$ $\mathbf{T}_i\mathbf{K}_i\mathbf{T}_i^T = \mathbf{K}'_i$			

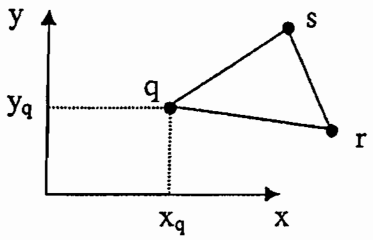
$a = L/AE$, AE = axial stiffness, $u = \cos\alpha$, $v = \sin\alpha$

Basic relationships, for element j :

displacements	$u = N^j d^j$
strains	$\varepsilon = B^j d^j$
stresses	$\sigma = D\varepsilon = D\, B^j d^j$
stiffness matrix	$K^j = \int (B^j)^T D\, B^j\, dV$
stiffness equations	$K^j d^j = p^j$

Material stiffness (for plane stress) $$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Shape functions of some simple plane stress elements

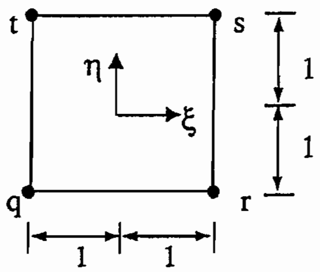


$n_q = [(x_r y_s - x_s y_r) + (y_r - y_s)x + (x_s - x_r)y]/2A$

$n_r = [(x_s y_q - x_q y_s) + (y_s - y_q)x + (x_q - x_s)y]/2A$

$n_s = [(x_q y_r - x_r y_q) + (y_q - y_r)x + (x_r - x_q)y]/2A$

A = area of triangle

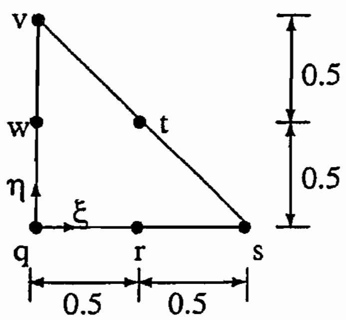


$n_q = (1-\xi)(1-\eta)/4$

$n_r = (1+\xi)(1-\eta)/4$

$n_s = (1+\xi)(1+\eta)/4$

$n_t = (1-\xi)(1+\eta)/4$



$n_q = (1-\xi-\eta)(1-2\xi-2\eta)$

$n_r = 4\xi\,(1-\xi-\eta)$

$n_s = \xi\,(2\xi-1)$

$n_t = 4\,\xi\,\eta$

$n_v = \eta\,(2\eta-1)$

$n_w = 4\eta\,(1-\xi-\eta)$