ENGINEERING TRIPOS PART IIA

Monday 7 May 2007 9 to 10.30

Module 3E3

MODELLING RISK

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Explain the purpose and use of Utility Theory. [20%]

(b) Explain the following terms: [3x4%]

(i) EMV

(ii) risk-neutral

(iii) CARA

(c) Define the term *insurance risk premium* and write down an approximation relating the insurance risk premium to the utility function and the yield. [10%]

(d) Assume that the yields are normally distributed with mean μ and variance σ^2. An investor uses the following utility

$$U(x) = -e^{-ax}$$

where a is a constant.
Calculate the utility of the expected yield and the expected utility. [30%]

(e) Find the insurance risk premium for this investor. [14%]

(f) Show that in this case the approximation from (c) holds exactly. [14%]

Hint: You might find the following integral useful:

$$\int_{-\infty}^{\infty} e^{-(Ax+B)^2+Cx} dx = \frac{\sqrt{\pi}}{A} \exp\left(\frac{C}{4A^2}(-4AB+C)\right)$$

2 (a) Write short notes on the following: [5x8%]

(i) Kendall's notation

(ii) utilization factor

(iii) exponential arrival times

(iv) steady state distribution

(v) Little's formula

(b) A small business has two phone operators, Alice and Bob. The system has limited capacity and any call that arrives when there are already 5 callers in the system is turned away. Each state is denoted by the number of customers n in the system and the following steady state probabilities are found:

n	P(n)
0	0.10
1	0.15
2	0.25
3	0.25
4	0.15
5	0.10

(i) What is the expected number of callers on hold? [10%]

(ii) What is the probability that a caller is turned away by the system? [5%]

(iii) If the caller arrival rate is 4 per hour, what is the expected total time that a caller will spend on the phone? [15%]

(iv) What is the utilization factor of the operators? [15%]

(v) What is the percentage of time that Alice or Bob are off the phone? [10%]

(vi) What is the percentage of time that both Alice and Bob are off the phone? [5%]

(TURN OVER

3 (a) Infinitopolis is an infinite city. All roads in the city are straight and infinite in length and run either in the North-South or the West-East direction. Each intersection either has a roundabout on it or a set of traffic lights. The intersections are close enough so that the neighbouring intersections are visible. A car drives in this infinite grid of roads. Which of the following can be modelled as a Markov chain in which the state represents the current intersection that the car is at? Justify your answers. [5x8%]

(i) At each intersection the car either drives straight, left or right with equal probability.

(ii) If a car arrives at a traffic light intersection, it drives straight, if it arrives at a roundabout intersection, it drives either left or right with equal probability.

(iii) If a car arrives at a traffic light intersection, it drives either North or South with equal probability, if it arrives at a roundabout it picks any of the four directions at random.

(iv) At each intersection the car drives North if it came from a traffic light intersection, otherwise it chooses a direction at random.

(v) At each intersection the car drives North if the intersection immediately to the North is a traffic light intersection. Otherwise it chooses a direction amongst West, South and East at random.

(b) Consider the following matrix:

$$M = \begin{pmatrix} 0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0.25 & 0.75 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0 & 0.5 \\ 0.25 & 0.25 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0 & 0.5 & 0 & 0.25 & 0 \\ 0 & 0.75 & 0 & 0 & 0 & 0.25 \end{pmatrix}$$

(i) Explain what is meant by a stochastic matrix. Is M stochastic? [10%]

(ii) Draw the transition network associated with M and determine the classes. [10%]

(iii) Starting from the initial vector $\mathbf{q} = (1,0,0,0,0,0)$ find the probability vector after two transitions. [10%]

(iv) Find the invariant distribution vector $\mathbf{u}$. Is $\mathbf{u}$ a limiting distribution? Justify your answer. [20%]

(cont.

(v) Find the states with the longest and shortest expected return times. [10%]

4 Assume you set up a simple linear regression model for pairs of observations (x_i, y_i)

$$y_i = \alpha + \beta x_i + \varepsilon_i$$

where all the ε_i are independent, identically-distributed, normal random variables with mean 0 and unknown variance σ^2. The x_i are chosen such that $\bar{x} = 0$.

(a) Working from first principles, derive least squares estimates a and b for the intercept and the slope respectively. [20%]

(b) Show that a and b are unbiased estimates for α and β respectively. [20%]

(c) Comment on the distributions of a and b. Find the variance of the slope estimate. [20%]

(d) Construct a $100(1-\gamma)\%$ confidence interval for the slope as a function of γ $(0 < \gamma < 1)$. [20%]

(e) Explain what is meant by a prediction interval and construct a $100(1-\gamma)\%$ prediction interval. [20%]

END OF PAPER