ENGINEERING TRIPOS PART IIA

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Tuesday 1 May 2007 9 to 10.30

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Module 3F1

SIGNALS AND SYSTEMS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

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1 The Tustin transformation can be used to transform systems between continuous-time and discrete-time. The transformation from a continuous-time $G(s)$ to a discrete-time $G(z)$ can be accomplished by replacing $s$ in $G(s)$ by:

$$s = \frac{2(z-1)}{T(z+1)}$$

where $T$ is the sampling period. One property about this transformation is that when applied to stable discrete-time systems it returns stable continuous-time systems and vice-versa. In this problem we consider $T = 2$.

(a) For $T = 2$, show that the inverse transformation is given by

$$z = \frac{1+s}{1-s}$$

[10%]

(b) Consider the discrete-time open-loop system

$$G(z) = \frac{(z+1)^2}{2z(z-1)}$$

(i) Use the Tustin transformation with $T = 2$ to obtain a continuous-time system $G(s)$. Draw the pole/zero diagram for both systems and conclude about their stability. [25%]

(ii) Consider the feedback system

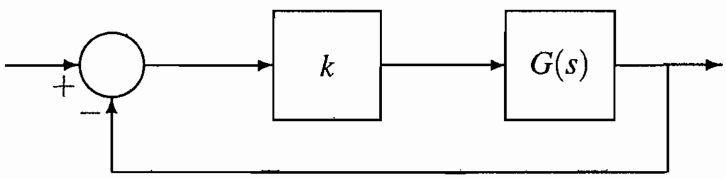


Fig. 1

where $k$ is a constant controller. Find the range of $k$ that results in stable feedback systems in Fig. 1 and infer about what ranges of $k$ stabilises $G(z)$. [25%]

(cont.

(c) With $T = 2$, prove the fact stated above, i.e. that the Tustin transformation when applied to stable discrete-time systems returns stable continuous-time systems and vice-versa (ie., $|z| < 1 \Leftrightarrow Re\{s\} < 0$) as follows:

(i) Let $s = \sigma + j\omega$ and find $|z|$. [20%]

(ii) Finally, show that $|z| < 1 \Leftrightarrow \sigma < 0$ (you need to show that $|z| < 1 \Rightarrow \sigma < 0$ and that $\sigma < 0 \Rightarrow |z| < 1$) . [20%]

(TURN OVER

2 (a) Assume a system has a single pole $p$ which is inside the unit disk:

$$G(z) = \frac{z}{z-p}$$

(i) Write down the system's pulse response $g_k$. [10%]

(ii) Write the convolution sum to express the output $y_k$ as a function of $\{g_i\}$ and the input $\{u_i\}$. [10%]

(iii) Show that

$$|y_k| \leq M \sum_{i=0}^{\infty} \left|p^i\right|$$

where $M$ is such that $|u_k| \leq M$ for all $k$. Hence show that the system is stable if $|p| < 1$. [30%]

(b) A random process, $X(t,\alpha)$, is a function of both time $t$ and a randomly chosen ensemble index $\alpha$.

(i) Define the autocorrelation function of such a process for sampling times, $t_1$ and $t_2$. Give the two conditions for $X(t,\alpha)$ to be Wide Sense Stationary (WSS). [20%]

(ii) Let

$$X(t,\alpha) = A(\alpha) \,.\, U(t)$$

where $A(\alpha)$ is a random process with a uniform pdf from -1 to 1 and $U(t)$ is a random Gaussian noise signal, with zero mean and autocorrelation function given by

$$r_{UU}(t_1,t_2) = e^{-|t_2-t_1|/T}$$

Find $r_{XX}(t_1,t_2)$, the autocorrelation function of $X(t,\alpha)$ and determine if $X(t,\alpha)$ is WSS. [30%]

3 (a) If a random variable $Y$ is the sum of two independent random variables $X_1$ and $X_2$, with pdfs $f_1(x)$ and $f_2(x)$, show that the conditional pdf of $Y$, given that $X_1 = x_1$, is

$$f(y|x_1) = f_2(y - x_1)$$

Hence show that $f_Y(y)$, the pdf of $Y$, may be expressed in terms of $f_1(x)$ and $f_2(x)$ using convolution. [30%]

(b) The characteristic function of $Y$ is defined in terms of an expectation as

$$\Phi_Y(u) = E[e^{juY}]$$

Express this in terms of $f_Y(y)$ and explain how the use of this function can simplify the analysis of sums of random processes with known pdfs. [30%]

(c) Let the pdfs of $X_1$ and $X_2$ be given by

$$f_1(x) = 1/b_1 \quad \text{if} \quad |x| < b_1/2 \qquad \text{and zero elsewhere.}$$

$$f_2(x) = \frac{1 - (|x|/b_2)}{b_2} \quad \text{if} \quad |x| < b_2 \qquad \text{and zero elsewhere.}$$

Calculate (using a Data Book) the characteristic functions of $X_1$ and $X_2$ and hence obtain the characteristic function of $Y$. [40%]

(TURN OVER

4 The entropy of a discrete random source $S$ with $N$ states is given by

$$H(S) = -\sum_{i=1}^{N} p_i \log_2(p_i)$$

where $p_i$ is the probability of state $i$.

(a) A first-order Markov random source emits symbols with pairwise joint probabilities given by the following table:

$P(X_n, X_{n+1})$ :

| $X_{n+1}$ \ $X_n$ | $A$ | $B$ | $C$ |
|---|---|---|---|
| $A$ | 0.72 | 0.04 | 0.04 |
| $B$ | 0.04 | 0.03 | 0.03 |
| $C$ | 0.04 | 0.03 | 0.03 |

Verify that this is a valid joint probability table, and calculate the mean probabilities of each of the symbols, $A$, $B$ and $C$. [15%]

(b) If the symbols are grouped in pairs for coding, design a Huffman code for encoding each symbol pair from the above source and calculate its efficiency with respect to the entropy of each pair. Why is this efficiency likely to be significantly better than that of a Huffman code for single symbols? [35%]

(c) Calculate the mutual information between adjacent symbols from the above source and the conditional entropy of each new symbol in the sequence (excluding the first). [35%]

(d) Briefly describe a coding scheme which could code this source at a bit rate per symbol approximately equal to the conditional entropy. [15%]

**END OF PAPER**

# Module 3F1, April 2007 – SIGNALS AND SYSTEMS – Solutions

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# Module 3F1, April 2007 – SIGNALS AND SYSTEMS – Answers

1 (b) (i) $G(s) = \frac{1}{s(s+1)}$. Both systems are (marginally) unstable.
(ii) $k > 0$.
(c) (i)

$$|z| = \frac{\sqrt{1+2\sigma+\sigma^2+\omega^2}}{\sqrt{1-2\sigma+\sigma^2+\omega^2}}$$

2 (a) (i) $g_k = p^k$, $k = 0, 1, 2, \ldots$.
(ii) $y_k = \sum_{i=0}^{k} u_{k-i} g_i = \sum_{i=0}^{k} u_{k-i} p^i$
(b) (ii) $r_{XX}(t_1, t_2) = \frac{1}{3} r_{UU}(t_1, t_2)$. The random process is WSS.

3 (b) $\Phi_Y(u) = \Phi_{X_1}(u)\Phi_{X_2}(u)$.
(c) $\Phi_{X_1}(u) = \text{sinc}(ub_1/2)$, $\Phi_{X_2}(u) = \text{sinc}^2(ub_2/2)$, $\Phi_Y(u) = \text{sinc}(ub_1/2)\,\text{sinc}^2(ub_2/2)$.

4 (a) $P(A) = 0.8$, $P(B) = 0.1$, $P(C) = 0.1$.
(b) Efficiency $= 91.92\%$.
(c) $I(X_{n+1}; X_n) = 0.1525$ and $H(X_{n+1}|X_n) = 0.7694$.

# Module 3F1, April 2007 – SIGNALS AND SYSTEMS – Answers

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(ii) $k > 0$.

(c) (i)

$$|z| = \frac{\sqrt{1+2\sigma+\sigma^2+\omega^2}}{\sqrt{1-2\sigma+\sigma^2+\omega^2}}$$

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3 (b) $\Phi_Y(u) = \Phi_{X_1}(u)\Phi_{X_2}(u)$.

(c) $\Phi_{X_1}(u) = \text{sinc}(ub_1/2)$, $\Phi_{X_2}(u) = \text{sinc}^2(ub_2/2)$, $\Phi_Y(u) = \text{sinc}(ub_1/2)\,\text{sinc}^2(ub_2/2)$.

4 (a) $P(A) = 0.8$, $P(B) = 0.1$, $P(C) = 0.1$.

(b) Efficiency $= 91.92\%$.

(c) $I(X_{n+1}; X_n) = 0.1525$ and $H(X_{n+1}|X_n) = 0.7694$.