ENGINEERING TRIPOS PART IIA

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Wednesday 2 May 2007 9 to 10.30

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Module 3F2

SYSTEMS AND CONTROL

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 (a) Given a linear system equation

$$\dot{x} = Ax + Bu$$

explain the role of the *matrix exponential function* $e^{At}$ in finding a solution $x(t)$, given an initial condition $x(0)$ and an input signal $u(t)$. [30%]

(b) If it is known that $W^{-1}AW = \Lambda$, and $\Lambda$ is a diagonal matrix, derive a relationship between $e^{At}$ and $e^{\Lambda t}$, and hence explain how $e^{At}$ can be evaluated. [30%]

(c) The small-perturbation longitudinal dynamics of an aircraft can be modelled by the state-space equation given in part (a) if the state variables are the forward speed, the normal (vertical) speed, and the pitch rate, and the input is the elevator angle, with

$$A = \begin{bmatrix} -0.1 & -0.01 & 0 \\ 0.02 & -2 & 400 \\ 0.01 & -0.2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.4 \\ -80 \\ -60 \end{bmatrix}.$$

Given that

$$W = \begin{bmatrix} -1 & 0.001j & -0.001j \\ -0.05 & 1 & 1 \\ -0.0002 & 0.02j & -0.02j \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} -1 & 0 & 0.05 \\ -0.025 & 0.5 & -25j \\ -0.025 & 0.5 & 25j \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & -2+8.9j & 0 \\ 0 & 0 & -2-8.9j \end{bmatrix},$$

evaluate the (3,1) element of $e^{At}$. What is the physical meaning of this element? [30%]

(d) Comment on the behaviour of the aircraft's transient response to a change of the elevator angle. (Explicit calculation of the response is not required.) [10%]

2 The attitude angle $\theta$ of a satellite (about one axis) is to be controlled by a pair of thrusters which exert a torque $u$. The transfer function from $u$ to $\theta$ is given by

$$\frac{\bar{\theta}(s)}{\bar{u}(s)} = \frac{s^2+a^2}{Js^2(s^2+b^2)}$$

with $a, b, J$ being real-valued positive parameters. The imaginary poles and zeros are due to fuel-sloshing effects (with damping so low that it can be neglected). The satellite's attitude is to be controlled by feedback from $\theta$ to $u$.

(a) If $a < b$, sketch the root-locus diagram if proportional negative feedback is used, and hence show that all the closed-loop poles are purely imaginary, for any value of proportional gain. [30%]

(b) If $a < b$ and phase-lead compensation is used:

$$\frac{\bar{u}(s)}{\bar{\theta}(s)} = -k\frac{s+z}{s+p} \quad \text{with} \quad 0 < z < p \quad \text{and} \quad k > 0$$

sketch the resulting root-locus diagram, and hence show that the closed loop is asymptotically stable for any positive value of $k$. [35%]

(c) If $a > b$ then phase-lead compensation as in part (b) may not be enough to obtain closed-loop stability. Show that the control law

$$\frac{\bar{u}(s)}{\bar{\theta}(s)} = -k\frac{s+z}{s+p} \times \frac{s^2+c^2}{s^2+d^2} \quad \text{with} \quad 0 < c < b < a < d, \quad 0 < z < p, \quad k > 0$$

ensures closed-loop asymptotic stability. [35%]

(TURN OVER

3 Figure 1 shows two masses connected to each other by a spring of stiffness $k$ and a damper with coefficient $c$. An actuator connected between the two masses exerts an equal and opposite force $u$ on each of them. The centre of mass $m_1$ is located at $y_1$, and that of mass $m_2$ is located at $y_2$. (The masses are constrained to move in one dimension only.) The system is initially at equilibrium with $u = y_1 = y_2 = 0$.

(a) Write down the equations of motion of the two masses, and put them into state-space form, using the state vector $x = [y_1, \dot{y}_1, y_2, \dot{y}_2]^T$ and taking $u$ as the input. [25%]

(b) Define the *controllability matrix* which is used to test for controllability of a state-space system, and state the controllability test. [25%]

(c) Suppose that $k = c = m_1 = m_2 = 1$. Show that the system is not controllable. [25%]

(d) With arbitrary masses $m_1$ and $m_2$, it can be shown that each column of the controllability matrix is orthogonal to the vector

$$[m_1, 0, m_2, 0]^T.$$

Explain how this is related to the fact that the combined centre of mass of the two masses remains fixed, whatever force $u$ the actuator exerts. [25%]

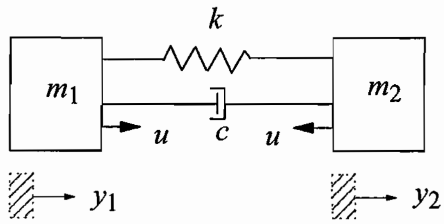


Fig. 1

4 A stability enhancement system for a car measures the yaw rate $r$ using a rate gyro, and changes the steering angle $\delta$. A simple model of the car is

$$\dot{x} = \begin{bmatrix} a_{11} & -1+c\sigma \\ c\sigma & a_{22} \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \delta,$$

where $x = [\beta, r]^T$ and $\beta$ is the sideslip angle. It is known that $a_{11} < 0$, $a_{22} < 0$, and $c > 0$.

(a) Show that this system is observable from $r$ providing that $\sigma \neq 0$. [25%]

(b) Assuming that $\sigma \neq 0$, design an observer which uses the measurement $r$ and has observer poles at $a_{11}$ and $a_{22}$. [25%]

(c) Draw a block-diagram showing how the observer output can be combined with a state feedback gain matrix. Explain how closed-loop stability can be ensured for such a system. [25%]

(d) What factors prevent the closed-loop response being made arbitrarily fast? [25%]

**END OF PAPER**

# Module 3F2: Systems and Control
## Answers 2007

1. (a) —
   (b) $e^{At} = We^{\Lambda t}W^{-1}$
   (c) $-0.0002e^{-0.1t} + 0.001e^{-2t}\sin(8.9t)$
   (d) —
2. —
3. (a)

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -c/m_1 & k/m_1 & c/m_1 \\ 0 & 0 & 0 & 1 \\ k/m_2 & c/m_2 & -k/m_2 & -c/m_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_2 \end{bmatrix} u$$

   (b) —
   (c) —
   (d) —
4. (a) —
   (b) $L = [-1 + c\sigma, \quad 0]^T$.
   (c) —
   (d) —