ENGINEERING TRIPOS PART IIA

Thurs 10 May 2007 2.30 to 4

Module 3F3

SIGNAL AND PATTERN PROCESSING

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) In a digital communication system random bits are transmitted as a Bernoulli random process, that is, each time point is independently assigned a value of +1 or -1 with equal probabilities of 0.5 . A possible sequence of bits generated from the process is, for example:

$$\{b_n\} = \{..., -1, +1, +1, -1, +1, +1, ...\}$$

Explain intuitively the meaning of stationarity for a discrete time random process. Define wide-sense stationarity. Show that the above random process is wide sense stationary. [30%]

(b) In the communications channel the bits are distorted according to a FIR filter,

$$x_n = \sum_{i=0}^{1} c_i b_{n-i}$$

where $c_0 = 1$ and $c_1 = 0.1$. Determine the cross-correlation function between $\{b_n\}$ and $\{x_n\}$, and also the autocorrelation function of $\{x_n\}$.

[40%]

(c) It is desired to optimally estimate the bit sequence $\{b_n\}$ from the channel data $\{x_n\}$. Design the second order FIR Wiener filter for this task, i.e. form an estimate of the type:

$$\hat{b}_n = \sum_{i=0}^{1} h_i x_{n-i}$$

where h_0 and h_1 are to be determined according to the Wiener criterion. [30%]

2 (a) Describe the principal means for reduction of errors in fixed precision digital filter implementations. Your description should include a discussion of overflow, limit cycles, saturation arithmetic and scaling. [30%]

(b) An IIR digital filter has the following transfer function:

$$H(z) = \frac{1-0.6z^{-1}}{1-0.9z^{-1}}$$

The filter is to be implemented in direct form II using 16-bit fixed point arithmetic. Determine appropriate scalings to ensure that overflow does not occur and that the full precision is used in internal calculations. The input signals to the system are *sine waves*. Sketch your implementation.

The input signal is assumed bounded between -1 and +1 and overflow occurs if the magnitude of the output or the internal signal exceeds 1. [30%]

(c) If the input to the above filter, with a scaling calculated as in part (b) above, is zero-mean Gaussian white noise with variance equal to 10, determine:

(i) the mean signal value at the output of the filter,

(ii) the power spectrum at the output of the filter,

(iii) the average power at the output of the filter, and hence

(iv) the probability that overflow occurs at any given sample time (you may assume that overflows occur only at the output of the filter, and that the effects of any previous overflows have died away).

[40%]

3 (a) Describe the steps involved in the window method of digital filter design. Explain its advantages and drawbacks when compared with the bilinear transform method of design. [30%]

(b) It is desired to design a lowpass digital filter having frequency response $D(\Omega)$. The ideal impulse response of the filter, d_n, is determined as the inverse DTFT of $D(\Omega)$:

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} D(\Omega)\exp(+j\Omega n)d\Omega$$

If d_n is truncated by multiplication with a finite duration window function w_n, show that the resulting filter's frequency response is

$$D_w(\Omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D(\lambda)W(\exp(j(\Omega-\lambda)))d\lambda$$

where $W(\exp(j\Omega))$ is the DTFT of the window function. Use this formula to explain qualitatively the effect on the filter's frequency response of truncating the ideal impulse response in this way. [30%]

(c) Instead of calculating d_n exactly using the inverse DTFT as above, the coefficients are estimated using the inverse DFT of a sampled version of $D(\Omega)$, as follows:

$$\hat{d}_n = \frac{1}{N}\sum_{p=0}^{N-1} D\left(\frac{2\pi p}{N}\right)\exp\left(\frac{j2\pi np}{N}\right)$$

Show that the resulting coefficients are related to the ideal coefficients by the following result:

$$\hat{d}_n = \sum_{m=-\infty}^{+\infty} d_{n-mN}$$

Explain how to reduce the effects of this approximation in a practical implementation of the window method of filter design. [40%]

[You may use the following result, which applies for integers m and k:

$$\sum_{p=0}^{N-1}\exp(j2\pi mp/N) = \begin{cases} N, & \text{for } m = kN \\ 0 & \text{otherwise} \end{cases}$$

(cont.

4 Assume you want to build an automatic berry classification machine which, based on the measured weight of the berry, x, classifies it into one of three classes, "Strawberry", "Raspberry" and "Cranberry", denoted $Y = s, Y = r$ and $Y = c$ respectively. Your goal is to compute $P(Y|x)$. Assume that $P(Y = s) = 0.5$, $P(Y = r) = 0.3$ and $P(Y = c) = 0.2$, which are obtained by measuring the observed frequencies of these berries.

(a) We make the approximation that $P(x|Y = s)$ is Gaussian with mean 4 and variance 1, that $P(x|Y = r)$ is Gaussian with mean 2 and variance 1, and that $P(x|Y = c)$ is Gaussian with mean 1 and variance 1. Calculate and sketch the class posterior probability for raspberries, i.e. $P(Y = r|x)$. Compute also the region of x for which the Raspberry class is more probable than the other two classes given x. Show all working. [30%]

(b) Given the following data set of 6 points and their class labels:

$$\mathscr{D} = \{(1.0, c), (2.0, c), (1.0, r), (3.0, r), (2.0, s), (4.0, s)\},$$

Derive the maximum likelihood parameter estimates of the means and variances of Gaussians fitting each class. [30%]

(c) Consider the problem of finding the means of the above 3 Gaussians which maximise the probability of the observed classes for the data set $\mathscr{D}$. The parameters of this model are $\theta = (\mu_s, \mu_r, \mu_c)$. Assume the variances are all set to one, and the observed class frequencies are set to $\pi_s = 0.5$, $\pi_r = 0.3$, $\pi_c = 0.2$ as above. We can write

$$P(Y|x, \theta) = \frac{\pi_Y e^{-(x-\mu_Y)^2/2}}{\pi_s e^{-(x-\mu_s)^2/2} + \pi_r e^{-(x-\mu_r)^2/2} + \pi_c e^{-(x-\mu_c)^2/2}}$$

and our goal is to maximize the probability of the observed classes for the above data set $\mathscr{D}$. Describe a method or algorithm for doing this. Would this algorithm give the same means as in part (b) of this question? Explain. [40%]

END OF PAPER