3A1 2008 Dr J Li

(1) Solution:

(a)

$$F(z) = -i\frac{\Gamma}{2\pi}\ln(z-ia) - i\frac{\Gamma}{2\pi}\ln(z+ia) = -i\frac{\Gamma}{2\pi}\ln\left((z-ia)(z+ia)\right) = -i\frac{\Gamma}{2\pi}\ln(z^2+a^2)$$

(b)

$$(u-iv) = \frac{dF}{dz} = -i\frac{\Gamma}{2\pi}\frac{2z}{z^2+a^2}$$

Stagnation points $\frac{dF}{dz} = 0$ only one at the origin.

(c) (i) The flow is unsteady since vortices move with the fluid – the induced velocity of one vortex on the other leads them to rotate.

(ii) The tangential velocity component is simply $\frac{\Gamma}{2\pi r}$ where r is the distance.

$$v_\theta = \frac{\Gamma}{2\pi 2a} = \frac{\Gamma}{4\pi a}$$

the angular velocity is

$$\frac{v_\theta}{r} = \frac{v_\theta}{a} = \frac{\Gamma}{4\pi a^2}$$

(check dimensions should be $1/T$)

$$\frac{d\beta}{dt} = \frac{\Gamma}{4\pi a^2}$$

(d) Sink at the origin

$$F(z) = \frac{m}{2\pi}\ln(z)$$

$$(u-iv) = \frac{dF}{dz} = \frac{m}{2\pi z} = \frac{m}{2\pi r}$$

directed inwards.

$$\frac{da}{dt} = v_r = \frac{m}{2\pi a}$$

Note from the outset that m is a negative number.

$$ada = \frac{m}{2\pi}dt$$

$$\frac{1}{2}a^2 = \frac{m}{2\pi}t + \frac{1}{2}a_0^2$$

where a_0 is a constant.

$$a = \sqrt{\frac{m}{\pi}t + a_0^2}$$

m is negative so a is reducing with time.

$$\frac{d\beta}{dt} = \frac{\Gamma}{4\pi a^2} = \frac{\Gamma}{4\pi(\frac{m}{\pi}t + a_0^2)}$$

$$\beta = \frac{\Gamma}{4\pi}\int \frac{1}{\frac{m}{\pi}t + a_0^2}dt = \frac{\Gamma}{4m}\int \frac{1}{t+\frac{a_0^2}{m}}dt = \frac{\Gamma}{4m}\ln(t+\frac{a_0^2}{m}) + c$$

say $\beta = 0$ when $t = 0$

$$\beta = \frac{\Gamma}{4m}\ln(\frac{mt}{\pi a_0^2}+1) \tag{1}$$

go back $a = \sqrt{\frac{m}{\pi}t + a_0^2} \rightarrow$ want to eliminate t. From Eqn. (1)

$$e^{4m\beta/\Gamma} = \frac{mt}{\pi a_0^2} + 1$$

$$t = -\frac{\pi a_0^2}{m}(e^{4m\beta/\Gamma} - 1)$$

$$a = \sqrt{a_0^2(e^{4m\beta/\Gamma}-1) + a_0^2} = a_0\sqrt{e^{4m\beta/\Gamma}} = a_0 e^{2m\beta/\Gamma}$$

where m is negative.

(2) Solution:

(a)

$$F(z) = Uz + \frac{\mu}{2\pi z}$$

$$\phi + i\psi = Ux + iUy + \frac{\mu(x-iy)}{2\pi(x^2+y^2)}$$

$$\psi = Uy - \frac{\mu y}{2\pi(x^2+y^2)} = Ur\sin\theta - \frac{\mu\sin\theta}{2\pi r}$$

where $r = a$, $r^2 = x^2 + y^2$, $y = r\sin\theta$. Want ψ =constant

$$\psi(r=a) = Ua\sin\theta - \frac{\mu\sin\theta}{2\pi a} = \frac{(2\pi Ua^2 - \mu)\sin\theta}{2\pi a}$$

(b) choose $\mu = 2\pi Ua^2$. $\psi = 0$ at $r = a$.

$$F(z) = Uz + \frac{2\pi Ua^2}{2\pi z} = Uz + \frac{Ua^2}{z^2}$$

stagnation points for interest

$$u - iv = \frac{dF}{dz} = U - \frac{Ua^2}{z^2} = 0$$

$$z = \pm a$$

(c)

$$F(z) = Uz + \frac{Ua^2}{z} + \frac{m}{2\pi}\ln(z+a) - \frac{m}{2\pi}\ln(z-a)$$

There are 4 stagnations points. To see this more clearly, find $\frac{dF}{dz} = 0$:

$$u - iv = \frac{dF}{dz} = U - \frac{Ua^2}{z^2} + \frac{m}{2\pi}\left(\frac{1}{z+a} - \frac{1}{z-a}\right) = 0$$

$$U(1-\frac{a^2}{z^2})(z^2-a^2)-\frac{ma}{\pi}=0$$

$$Uz^4-\left(\frac{ma}{\pi}+2Ua^2\right)z^2+Ua^4=0$$

4^{th} order polynomial 4 roots. By considering the symmetry and looking at the flow pattern we can guess they are all on the real axis.

Let $\zeta = z^2$

$$\zeta^2-\left(\frac{ma}{\pi U}+2a^2\right)\zeta+a^4=0$$

$$\zeta = a^2\left[(1+\frac{m}{2\pi aU})\pm\frac{m}{2\pi aU}\sqrt{1+\frac{4\pi aU}{m}}\right]$$

$z^2=\zeta$, since m has been chosen as a magnetude then it's a positive number. Hence there are 4 real roots. (note should be symmetrical about $x=0$). $U=1$, $a=1$:

$$m=2\pi aU, \quad \zeta=a^2(2\pm\sqrt{3}), \quad z=\pm\sqrt{\zeta}$$

(3) Solution:

(a)

$$\frac{U}{U_\infty}=1-e^{-\eta}$$

$$\theta=\int\frac{U}{U_\infty}\left(1-\frac{U}{U_\infty}\right)dy=\delta^*\int(1-e^{-\eta})e^{-\eta}d\eta=\delta^*\int_0^\infty e^{-\eta}-e^{-2\eta}d\eta$$

$$=\delta^*\left[-e^{-\eta}+\frac{1}{2}e^{-2\eta}\right]_0^\infty=\frac{\delta^*}{2}$$

$$H=\frac{\delta^*}{\theta}=2$$

$$\tau_w=\mu\frac{\partial u}{\partial y}|_{y=0}=\mu\frac{\partial}{\partial y}\left(U_\infty(1-e^{-\eta})\right)|_{y=0}=\mu\frac{\partial}{\partial\eta}\left(U_\infty(1-e^{-\eta})\right)\frac{\partial\eta}{\partial y}|_{y=0}=\mu U_\infty e^{-\eta}\frac{1}{\delta^*}_{\eta=0}=\frac{\mu U_\infty}{\delta^*}$$

where $y=0\Rightarrow\eta=0$

$$C_f'=\frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}=\frac{2\nu}{U_\infty\delta^*}$$

Momentum integral equation

$$\frac{d\theta}{dx}+\frac{\theta}{U_\infty}\frac{dU_\infty}{dx}(H+2)=\frac{C_f'}{2}$$

(b)

$$\frac{1}{2}\frac{d\delta^*}{dx}+\frac{\delta^*}{2U_\infty}\frac{dU_\infty}{dx}(4)=\frac{\nu}{U_\infty\delta^*}$$

(c)

$$\frac{dU_\infty}{dx}=0$$

$$\frac{d\delta^*}{dx} = \frac{2\nu}{U_\infty \delta^*}$$

$$\delta^* \frac{d\delta^*}{dx} = \frac{2\nu}{U_\infty}$$

$$\frac{1}{2}\left(\delta^*\right)^2 = \frac{2\nu}{U_\infty}x + \text{constant}$$

Say $\delta^* = 0$ at $x = 0$,

$$\delta^* = 2\sqrt{\frac{\nu x}{U_\infty}}$$

(d) want constant δ^*, $\frac{d\delta^*}{dx} = 0$

$$\frac{2\delta^*}{U_\infty}\frac{dU_\infty}{dx} = \frac{\nu}{U_\infty \delta^*}$$

$$2\delta^* \frac{dU_\infty}{dx} = \frac{\nu}{\delta^*}$$

$$\frac{dU_\infty}{dx} = \frac{\nu}{2\left(\delta^*\right)^2}$$

$$U_\infty = \frac{\nu}{2\left(\delta^*\right)^2}x + \text{constant}$$

Not necessary to find the constant.

(4) Solution:

(a) Draw a circuit along the plate. Since $v = 0$ then the ends make no contributions and also if the top of the circuit is far from the plate then it makes no contribution so

$$\Gamma = U \cdot 1 = U$$

(b) Stokes theorem:

$$\oint_c \mathbf{u} \cdot d\mathbf{l} = \int_A \omega \cdot d\mathbf{S}$$

where c is a closed circuit and A is the area enclosed. Now in our case $dS = dy \cdot 1$ So

$$\Gamma(\text{per unit length}) = \int_0^\infty \omega dy = \omega_0 \int_0^\infty e^{\frac{-y^2}{4\nu t}} dy = \omega_0\sqrt{4\nu t}\int_0^\infty e^{\frac{-y^2}{4\nu t}} d\left(\frac{y}{\sqrt{4\nu t}}\right) = \omega_0\sqrt{\pi\nu t}$$

(c) Now from (a) $\Gamma = U$, so $U = \omega_0\sqrt{\pi\nu t}$.

$$\omega_0 = \frac{U}{\sqrt{\pi\nu t}}$$

(d)

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y}$$

Now wall shear stress

$$\tau_w = \mu\frac{\partial u}{\partial y}|_{y=0} = -\mu\omega_0 = -\frac{\mu U}{\sqrt{\pi\nu t}}$$

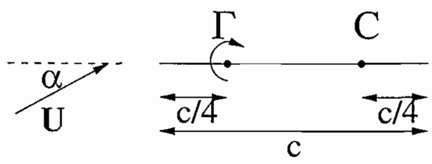


Figure 1: 2D lumped-parameter model

$$\text{Force} = \tau_w \times \text{wall area} = \tau_w \times 1 \times 1$$

(5) Solution:

(a) Flow in which the vorticity is zero. It is a good approximation in many flow situations or regions and it reduces the Euler equations to a linear set of equations. In this way solutions can be added.

(b) In the absence of viscosity there are only pressure forces which act through the centre-of-gravity of fluid elements – thus they cannot exert any torque and so cannot start an element spinning (no torque - no angular acceleration).

(c) The only sources of vorticity are boundaries or surfaces since this is the only place a net torque can be exerted on the fluid.

(d) This term is a source term in which a torque can be applied by a pressure gradient acting on a region of non-uniform density. The simplest interpretation is that the variation in density shifts the center of gravity of an element. And hence the pressure exerts a torque.

(e) Vortex stretching is the stretching of vortex lines by a strain field. A simple example is swirling flow through a contraction in a pipe but there are many others – it is important in turbulent flows. It is also important in the horse-shoe vortex system in front of a wall mounted pylon.

Γ = constant for a loop around by Kelvin.

$$\Gamma = \text{const.} = \int \omega \cdot dS = \int \frac{\omega}{l} = \frac{\omega_0}{l} = \text{const.}$$

$$\text{Vol} = dS \cdot l \ \ (\text{conservationofvolume})$$

$$dS = \frac{constant}{l}$$

Vortcity is proportional to length.

(f) As the vortex is stretched gradients in vorticity increase leading to greater diffusion which tends to spread the vorticity.

(6) Solution:

(a) 2D lumped-parameter model (Fig. 1):

- Point vortex, circulation Γ, at quarter-chord point.

- Collocation point C at three-quarters chord.

- Γ determined by normal velocity b.c. at C. In the absence of external influences, this is $U\alpha = \Gamma/\pi c$.

(b)

$$\frac{dy_c}{dx} = \frac{h}{c}\left[1 - 4\frac{x}{c} + 3\left(\frac{x}{c}\right)^2\right]$$

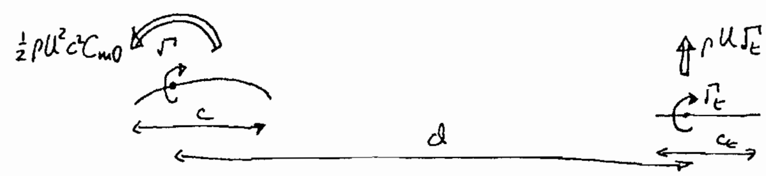
Express in term of θ, where $\frac{2x}{c} = 1 + \cos\theta$. After some manipulation:

$$\frac{dy_c}{dx} = \frac{h}{c}\left[\frac{1}{8} - \frac{1}{2}\cos\theta + \frac{3}{8}\cos 2\theta\right];$$

$$-2\frac{dy_c}{dx} = \frac{h}{c}\left[-\frac{1}{4} + \cos\theta - \frac{3}{4}\cos 2\theta\right];$$

i.e. in standard Fourier series (data card): $g_0 = -\frac{1}{4}\frac{h}{c}$, $g_1 = \frac{h}{c}$, $g_2 = -\frac{3}{4}\frac{h}{c}$.

(i) $c_l = \pi(g_0 + g_1/2) = \frac{\pi}{4}\frac{h}{c}$ (ii) $c_{\mathrm{m0}} = \frac{\pi}{8}(g_0 + g_1) = \frac{\pi}{32}\frac{h}{c}$



(c) Overall pitching moment:

$$\frac{1}{2}\rho U^2 c^2 c_{\mathrm{mD}} + \rho U \Gamma_t d = 0$$

$$\Gamma_t = -\frac{\pi}{64}\frac{Uch}{d}$$

N.B. $\rho U\Gamma = \frac{1}{2}\rho U^2 c c_l \Rightarrow \Gamma = \frac{\pi}{8}Uh$

Collocation point b.c. is

$$\frac{\Gamma_t}{\pi c_t} + \frac{\Gamma}{2\pi d} = U\alpha_t \Rightarrow \alpha_t = \frac{1}{16}\frac{h}{d}\left(1 - \frac{c}{4c_t}\right)$$

(7) Solution:

(a) Wing lift $= \rho U \int_{-s}^{s}\Gamma(y)dy = \frac{\rho U^2 s^2}{10}\int_{-1}^{1}(1 - y^2)dy = \frac{2}{15}\rho U^2 s^2$

(b) (i) if $\Gamma(y) = Us\sum_{\mathrm{n\ odd}} G_n \sin n\theta$, then $\int_0^{\pi}\Gamma(y)\sin m\theta d\theta = \frac{\pi}{2}UsG_m$.

So $G_m = \frac{2}{\pi Us}\int_0^{\pi}\Gamma(y)\sin m\theta d\theta = \frac{1}{5\pi}\int_0^{\pi}\sin^2\theta \sin m\theta d\theta$ (orthorgonality of sins). Hence $G_1 = \frac{4}{15\pi}$, $G_3 = -\frac{4}{75\pi}$.

(ii) For the elliptical distribution $C_{\mathrm{Di}} = \frac{C_L^2}{\pi AR}$.

For our wing $C_{\mathrm{Di}} = (1+\delta)\frac{C_L^2}{\pi AR}$ with $\delta \approx 1 + \left(\frac{G_3}{G_1}\right)^2$. $G_3/G_1 = -1/5 \Rightarrow \delta \approx 1.04$, ie, a 4 % excess over elliptical.

(c) $c_l(y) \propto \Gamma(y)/c(y)$, so $dc_l/dy \propto \frac{cd\Gamma/dy - \Gamma dc/dy}{c^2}$.

Max c_l at $cd\Gamma/dy - \Gamma dc/dy = 0$;

$$(1 - \frac{1}{2}y)[-2y] - (1 - y^2)(-\frac{1}{2}) = 0$$

$$\frac{1}{2}y^2 - 2y + \frac{1}{2} = 0$$

$$y = 2 - \sqrt{3} = 0.268$$

Implications: Stall not at root (ideal location to avoid danger of spin), but far enough from tip to represent acceptable compromise.

(8) solution:

(a) The wind tunnel has no contraction and suffered from flow quality. In particular it had thick side-wall layers (giving poor flow uniformity) as well as a high level of free-stream turbulence.

(b)The design can be improved through the addition of a contraction. This implies that a diffuser is fitted after the working section and that turbulence screens and honeycombs are placed in the settling chamber (the large cross-section region ahead the contraction).

Contractions introduce a strong favourable streamwise pressure gradient. This makes side-wall boundary layers much thinner and thus improves flow uniformity. The rapid acceleration also considerably reduce free-stream turbulence levels through vortex stretching. Screens help to reduce turbulence levels even further and also improve flow uniformity (note that Prandtl's original design feature screens).

(c) The introduction of a contraction and diffuser significantly reduces the flow speed elsewhere in the tunnel (compared to the working section). Since frictional losses scale with velocity cubed, this has a big effect on the power requirement because most of the loss is generate in turning vanes and screens (all of which are located in low speed regions). The addition of screens, however, increase the power requirement, but since screens were also part of the original design the overall effect of the contraction is to reduce the power requirement.

A downside of this change is that either the working section is much smaller or the space needed by the wind tunnel is much greater.

(d) Flow direction can be measure with various multi-hole probes, crossed hot-wires and also by laser flow diagnostics. The advantages and drawbacks of these techniques are all explained in the lecture hand-out.