1 (a) (b) Think of cubes as AAA bodies that, like spheres, have all axes through their centre principal.

The body described is an 8-cube cube with a cube missing.

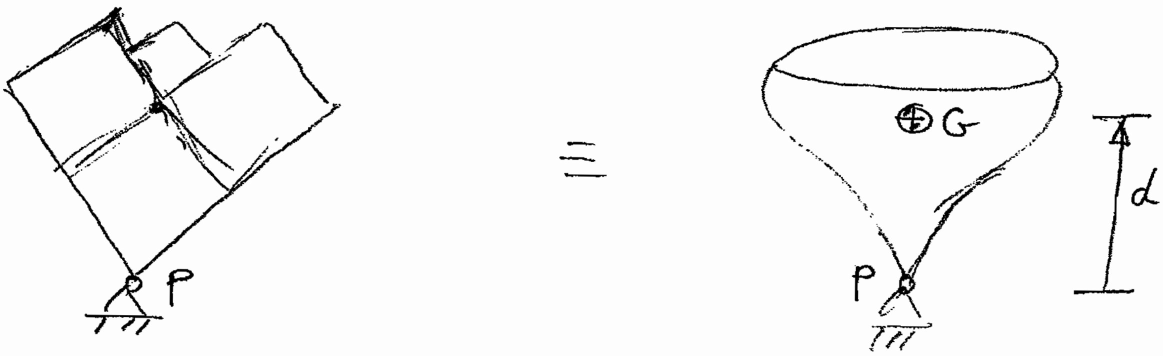
For the small cube $A_1 = \frac{1}{6} m a^2$

For the eight-cube cube $A_8 = \frac{1}{6} 8m (2a)^2 = \frac{32}{6} ma^2$

The moment of inertia along PO is, since PO is principal for both cubes,

$$C = A_8 - A_1 = \frac{31}{6} ma^2$$ [25%]

The resulting 7-cube body is an AAC body, just like a spinning top. So we can draw it like a top



[25%]

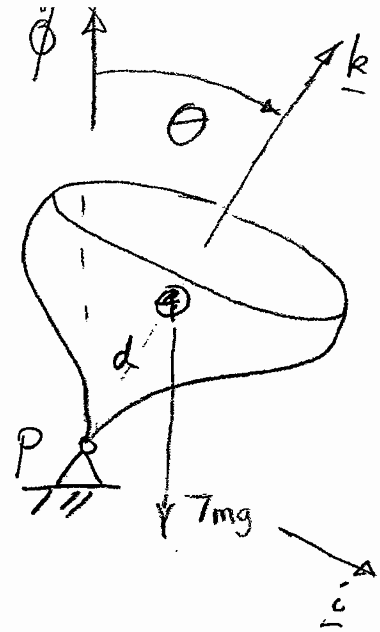
(b) G is distance r from O such that the "missing" cube of mass m when added to the 7m body has its G at O

hence $7m\, r = m \frac{a}{2}\sqrt{3}$

$\therefore r = \frac{a}{14}\sqrt{3}$

Hence distance $PG = d = a\sqrt{3}\left(1 - \frac{1}{14}\right) = \frac{13\sqrt{3}\, a}{14}$ [25%]

(d) Steady precession of a fast-spinning top



Gyro equation 2

$$-C\omega_3\omega_1 = Q_2$$

with $Q_2 = 7mgd\sin\theta$

and $\omega_1 = -\dot{\phi}\sin\theta$ and $\omega_3 = \omega$

$$\therefore \quad \dot{\phi} = \frac{7mgd}{C\omega}$$

and $C = \frac{31}{6}ma^2$

$$d = \frac{13\sqrt{3}}{14}a$$

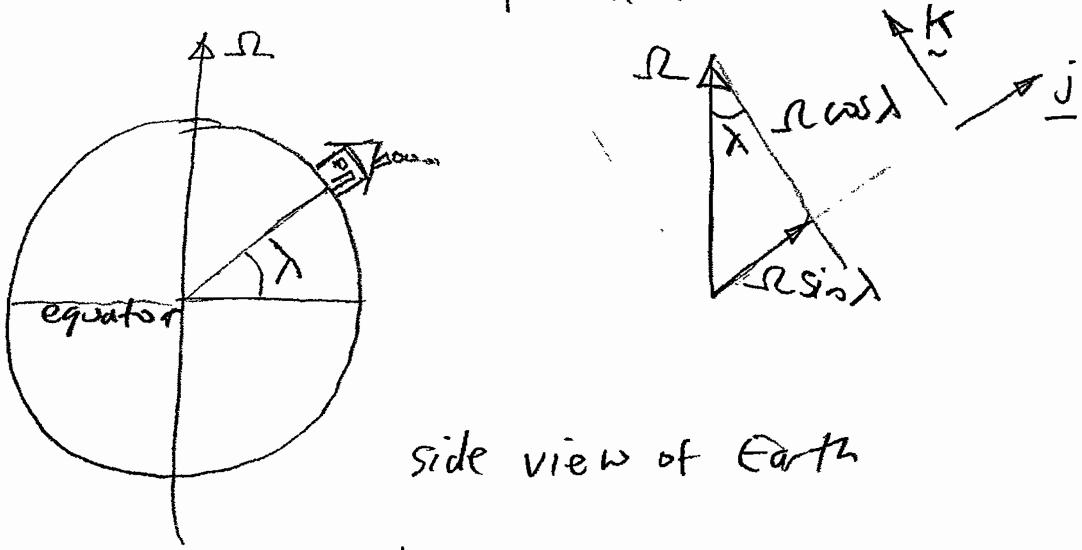
$$\therefore \quad \dot{\phi} = \frac{13\sqrt{3}}{14}\,\frac{6}{31}\,\frac{7g}{a\omega}$$

$$= \frac{78\sqrt{3}}{62}\,\frac{g}{a\omega}$$

[25%]

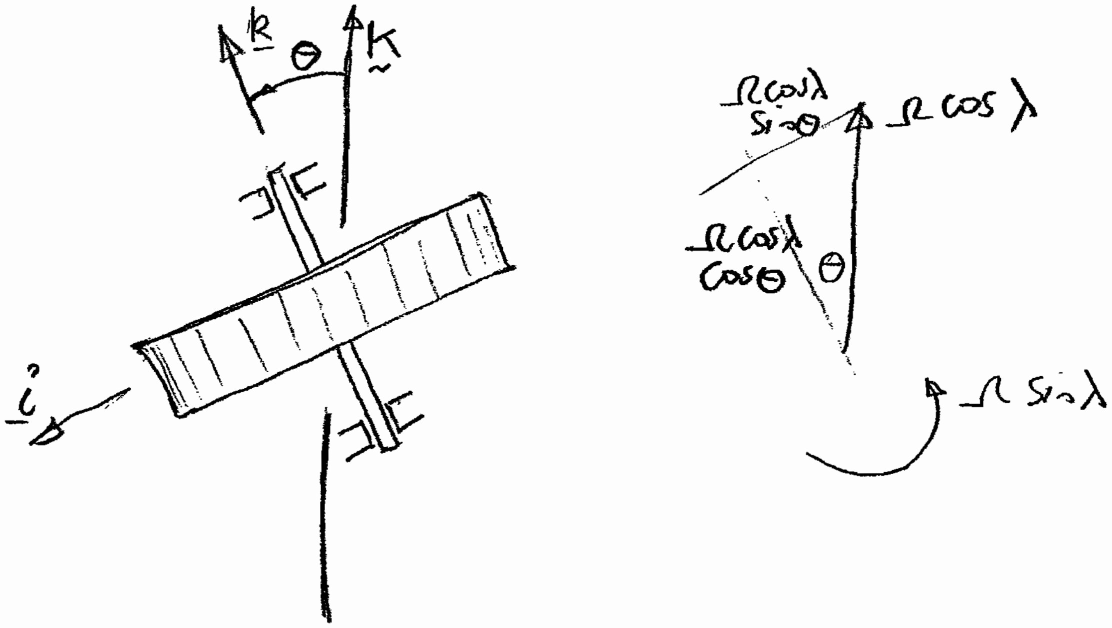
$$= 2.18\,\frac{g}{a\omega}$$

2 (a) First resolve the Earth's Ω into the local reference frame



side view of Earth

Next resolve the $\Omega \cos\lambda$ component into its $\underline{i}$ & $\underline{k}$ components

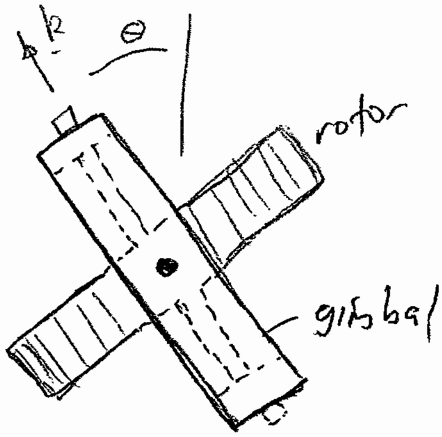


So the angular velocity of the reference frame (the gimbal) is

$$\Omega_1 = -\Omega \cos\lambda \sin\theta$$
$$\Omega_2 = \Omega \sin\lambda + \dot{\theta}$$
$$\Omega_3 = \Omega \cos\lambda \cos\theta$$

and with fast spin $\omega_3 \gg \Omega_3$ [30%]

(b) The gimbal is free to
(c) rotate about the $\underline{j}$ axis
and the rotor is free
about the $\underline{k}$ axis but
the $\underline{i}$ axis is constrained



So $Q_1 \neq 0 \qquad Q_2 = 0 \ , \ Q_3 = 0$

Use the second gyro equation

$$A\dot{\Omega}_2 + (\cancel{A\Omega_3} - C\omega_3)\Omega_1 = 0$$

(fast spin)

$$\therefore \quad A\ddot{\theta} + C\omega_3 \Omega \cos\lambda \sin\theta = 0$$

$$\therefore \quad \ddot{\theta} + \lambda^2 \sin\theta = 0 \qquad \lambda^2 = \frac{C\omega_3 \Omega \cos\lambda}{A}$$

This has two steady-state solutions for $\sin\theta = 0$

$$\theta = 0 \quad , \quad \theta = \pi$$

Test these for stability :

θ = small $\quad \therefore \quad \ddot{\theta} + \lambda^2 \theta = 0$

Stable SHM at frequency λ [40%]

$\theta = \pi + \alpha \quad , \quad \sin\theta = -\sin\alpha \quad , \ \ddot{\theta} = \ddot{\alpha}$

α = small $\quad \therefore \quad \ddot{\alpha} - \lambda^2 \alpha = 0$

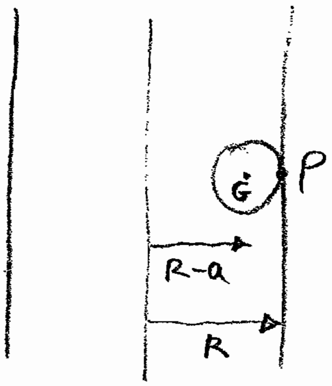
unstable $e^{\lambda t}$ solutions

Hence the gyrocompass oscillates about
$\theta = 0$ (ie North)

Damping causes this oscillation to die away
and the gyrocompass comes to rest
pointing north

[30%]

3



(a) P in the ball is at rest (no slip) and G is moving on a horizontal circle

$$\underline{v}_G = (R-a)\,\Omega\,\underline{j}$$

$$\underline{v}_P = \underline{v}_G + \underline{\omega} \times a\,\underline{i}$$

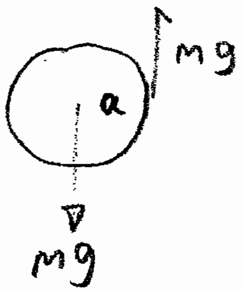
$$= (R-a)\Omega\,\underline{j} + \omega_3 a\,\underline{j} - \omega_2 a\,\underline{k}$$

$$= 0$$

$$\therefore \quad \omega_3 = -\frac{(R-a)}{a}\Omega \qquad \text{and} \quad \omega_2 = 0 \qquad [25\%]$$

No constraint on ω_1 yet.

(b)



(i) $\underline{Q} = -mga\,\underline{j}$

(ii) $\underline{h} = A(\omega_1\underline{i} + \omega_2\underline{j} + \omega_3\underline{k})$

$$= \tfrac{2}{5}ma^2\left(\omega_1\underline{i} - \frac{R-a}{a}\Omega\,\underline{k}\right)$$

[25%]

(c) $\underline{Q} = \dot{\underline{h}}$ and in steady state $\omega_1 = \text{const}$, $\Omega = \text{const}$

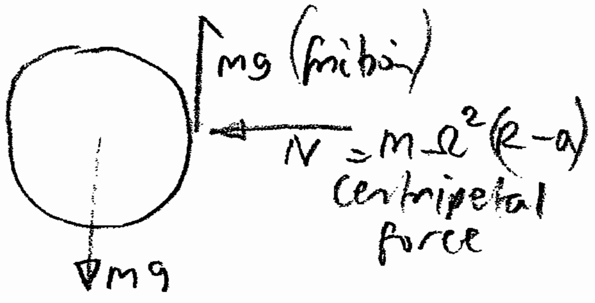
$$\therefore \dot{\underline{h}} = \dot{\underline{h}}\big|_{rot} + \underline{\Omega}\times\underline{h}$$

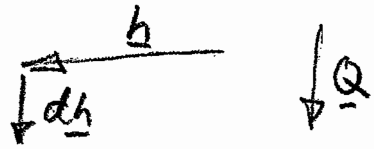
$$= 0 + \Omega\underline{k}\times\underline{h}$$

$$= \tfrac{2}{5}ma^2\omega_1\Omega\,\underline{j}$$

$$\therefore -mga\,\underline{j} = \tfrac{2}{5}ma^2\omega_1\Omega\,\underline{j} \qquad \therefore \underline{\underline{\omega_1 = -\frac{5}{2}\frac{g}{a\Omega}}} \qquad [25\%]$$

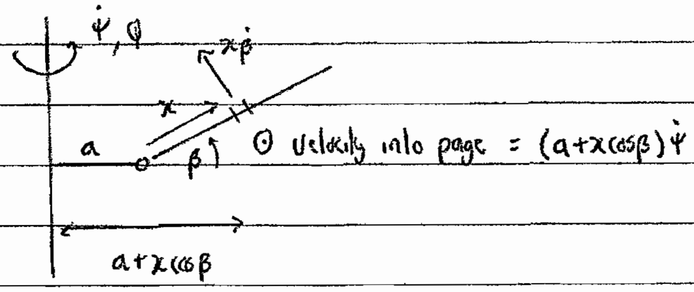
(d)





top view [25%]

4) a)



$$T = \tfrac{1}{2} m \int_0^L \left[(x\dot{\beta})^2 + (a + x\cos\beta)^2 \dot{\psi}^2 \right] dx$$

$$\underline{T = \tfrac{1}{6} m L^3 \dot{\beta}^2 + \tfrac{1}{2} m a^2 L \dot{\psi}^2 + \tfrac{1}{2} m a L^2 \cos\beta\, \dot{\psi}^2 + \tfrac{1}{6} m L^3 \cos^2\beta\, \dot{\psi}^2}$$ [25%]

b) For β, generalized momentum $\equiv \dfrac{\partial T}{\partial \dot{\beta}} = \underline{\tfrac{1}{3} m L^3 \dot{\beta}}$ = moment of momentum about flap hinge

For ψ, generalized momentum $\equiv \dfrac{\partial T}{\partial \dot{\psi}} = \underline{(m a^2 L + m a L^2 \cos\beta + \tfrac{1}{3} m L^3 \cos^2\beta)\dot{\psi}}$

= moment of momentum about rotor axis [25%]

c) For β, $\dfrac{d}{dt}\left[\dfrac{\partial T}{\partial \dot{\beta}}\right] - \dfrac{\partial T}{\partial \beta} + \dfrac{\partial V}{\partial \beta} = 0$ with $V = (mL) g \times \tfrac{1}{2} L \sin\beta$ ← gravitational P.E.

$$\Rightarrow \tfrac{1}{3} m L^3 \ddot{\beta} - \left[-\tfrac{1}{2} m a L^2 \sin\beta\, \dot{\psi}^2 - \tfrac{1}{3} m L^3 \cos\beta . \sin\beta\, \dot{\psi}^2 \right] + \tfrac{1}{2} m g L^2 \cos\beta = 0$$

$$\Rightarrow \underline{\tfrac{1}{3} m L^3 \ddot{\beta} + \left[\tfrac{1}{2} m a L^2 \sin\beta + \tfrac{1}{3} m L^3 \cos\beta \sin\beta \right] \dot{\psi}^2 = -\tfrac{1}{2} m g L^2 \cos\beta} \quad \text{— } (*)$$

For ψ, $\dfrac{d}{dt}\left[\dfrac{\partial T}{\partial \dot{\psi}}\right] - \dfrac{\partial T}{\partial \psi} + \dfrac{\partial V}{\partial \psi} = 0$

$$\Rightarrow \underline{\frac{d}{dt}\left[(m a^2 L + m a L^2 \cos\beta + \tfrac{1}{3} m L^3 \cos^2\beta)\dot{\psi} \right] = 0}$$ [25%]

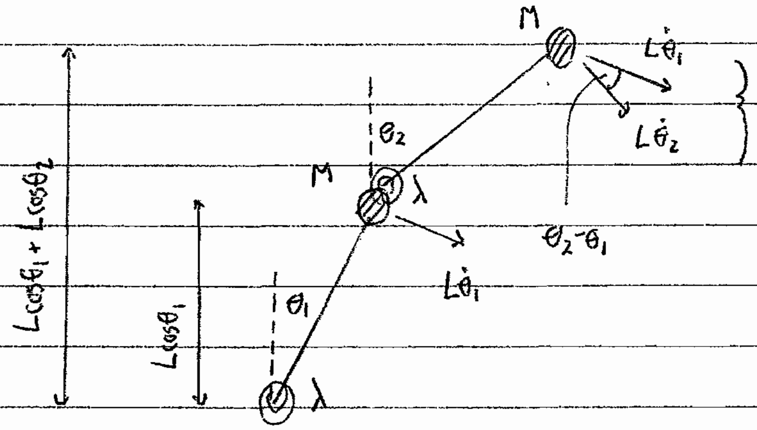
↑ could be expanded

d) For $\dot{\psi}$ = constant, and β small, equation $(*)$ becomes:

$$\tfrac{1}{3} m L^3 \ddot{\beta} + \left(\tfrac{1}{2} m a L^2 + \tfrac{1}{3} m L^3 \right) \dot{\psi}^2 \beta = -\tfrac{1}{2} m g L^2$$ [25%]

For SHM, $\underline{\omega_n^2} = \left(\tfrac{1}{2} m a L^2 + \tfrac{1}{3} m L^3\right)\dot{\psi}^2 / \left(\tfrac{1}{3} m L^3\right) = \underline{\dot{\psi}^2 \left(1 + \tfrac{3a}{2L}\right)}$

5 a)



Resultant velocity^2 from cosine rule

$= (L\dot\theta_1)^2 + (L\dot\theta_2)^2 + 2L^2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)$

$$T = \tfrac{1}{2}M\left[(L\dot\theta_1)^2 + (L\dot\theta_1)^2 + (L\dot\theta_2)^2 + 2L^2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)\right]$$

$$V = Mg\left[L\cos\theta_1 + L\cos\theta_1 + L\cos\theta_2\right] + \tfrac{1}{2}\lambda\theta_1^2 + \tfrac{1}{2}\lambda(\theta_2-\theta_1)^2$$ [25%]

b) For mass matrix $M_{ij} = \dfrac{\partial^2 T}{\partial\dot\theta_i\partial\dot\theta_j}\Big|_{\theta_i=\theta_j=0}$

$$\frac{\partial^2 T}{\partial\dot\theta_1^2}\Big|_0 = 2ML^2 \qquad \frac{\partial^2 T}{\partial\dot\theta_1\partial\dot\theta_2}\Big|_0 = ML^2 \qquad \frac{\partial^2 T}{\partial\dot\theta_2^2} = ML^2$$

For stiffness matrix $k_{ij} = \dfrac{\partial^2 V}{\partial\theta_i\partial\theta_j}\Big|_{\theta_i=\theta_j=0}$

$$\frac{\partial^2 V}{\partial\theta_1^2}\Big|_0 = -2MgL + 2\lambda \qquad \frac{\partial^2 V}{\partial\theta_1\partial\theta_2}\Big|_0 = -\lambda \qquad \frac{\partial^2 V}{\partial\theta_2^2}\Big|_0 = -MgL + \lambda$$

$$\Rightarrow ML^2\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{pmatrix} + \begin{pmatrix} -2MgL+2\lambda & -\lambda \\ -\lambda & -MgL+\lambda\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$ [30%]

c) For natural frequencies $|-\omega^2 M + k| = 0$

$$\Rightarrow \begin{vmatrix} -2ML^2\omega^2 - 2MgL + 2\lambda & -ML^2\omega^2 - \lambda \\ -ML^2\omega^2 - \lambda & -ML^2\omega^2 - MgL + \lambda \end{vmatrix} = 0$$

$$\Rightarrow 2(ML^2\omega^2 + MgL - \lambda)^2 = (ML^2\omega^2 + \lambda)^2 \Rightarrow (ML^2\omega^2 + MgL - \lambda) = \pm\frac{1}{\sqrt2}(ML^2\omega^2 + \lambda)$$ — [*]

For lowest frequency take "−" root $\Rightarrow$ $(1 + \frac{1}{\sqrt2})ML^2\omega^2 = (1 - \frac{1}{\sqrt2})\lambda - MgL$

This yields $\omega = 0$ for $MgL/\lambda = 1 - \frac{1}{\sqrt{2}}$

For $MgL/\lambda > 1 - \frac{1}{\sqrt{2}}$ the structure is unstable and will buckle [20%]

d) Neglecting "g" terms, $\left(1+\frac{1}{\sqrt{2}}\right)ML^2\omega^2 = \left(1-\frac{1}{\sqrt{2}}\right)\lambda \Rightarrow \omega_1^2 = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\frac{\lambda}{ML^2}$

or $\left(1-\frac{1}{\sqrt{2}}\right)ML^2\omega^2 = \left(1+\frac{1}{\sqrt{2}}\right)\lambda \Rightarrow \omega_2^2 = \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\frac{\lambda}{ML^2}$

For mode shapes: $\begin{pmatrix} -2ML^2\omega^2 + 2\lambda & -ML^2\omega^2 - \lambda \\ \checkmark & \checkmark \end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} = \begin{pmatrix} 0 \\ \checkmark \end{pmatrix}$

And $2ML^2\omega^2 - \lambda 2 = \overset{\text{mode 2}}{\underset{\text{mode 1}}{\pm}} \sqrt{2}\,(ML^2\omega^2 + \lambda)$ from equation [*]

$\Rightarrow$ Mode 1 has $(-\sqrt{2}\,\theta_1 + \theta_2) = 0 \Rightarrow \theta_2 = \sqrt{2}\,\theta_1$

Mode 2 has $(+\sqrt{2}\,\theta_1 + \theta_2) = 0 \Rightarrow \theta_2 = -\sqrt{2}\,\theta_1$ [25%]