Prof J Woodhouse

# Part IIA Module 3C6 2008 : crib

1(a) From data sheet, $EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = 0$

So if $w = u(x)e^{i\omega t}$, $u'''' = \alpha^4 u$ where $\alpha^4 = \frac{m\omega^2}{EI}$

General solution $u = A\cos\alpha x + B\sin\alpha x + C\cosh\alpha x + D\sinh\alpha x$

At $x=0$: $\begin{cases} w=0 \rightarrow A + C = 0 \\ w' = 0 \rightarrow \alpha(B+D) = 0 \end{cases}$

At $x=L$: $w=0 \rightarrow A\cos\alpha L + B\sin\alpha L - A\cosh\alpha L - B\sinh\alpha L = 0$

$w'=0 \rightarrow \alpha\{-A\sin\alpha L + B\cos\alpha L - A\sinh\alpha L - B\cosh\alpha L\} = 0$

For non trivial solution ($A, B$ not both zero) need $\det[\because] = 0$

$\therefore (\cos\alpha L - \cosh\alpha L)(\cos\alpha L - \cosh\alpha L) = (\sin\alpha L - \sinh\alpha L)(-\sin\alpha L - \sinh\alpha L)$

$\therefore \cos^2\alpha L - 2\cos\alpha L\cosh\alpha L + \cosh^2\alpha L = \sinh^2\alpha L - \sin^2\alpha L$

$\therefore 2\cos\alpha L\cosh\alpha L = 2$, ie $\cos\alpha L\cosh\alpha L = 1$

(b) Start as above, but different boundary condition at $x=L$:

$\begin{cases} w = 0 \rightarrow A\cos\alpha L + B\sin\alpha L - A\cosh\alpha L - B\sinh\alpha L = 0 \\ w'' = 0 \rightarrow \alpha^2\{-A\cos\alpha L - B\sin\alpha L - A\cosh\alpha L - B\sinh\alpha L\} = 0 \end{cases}$

So $\det[\because] = 0 \rightarrow$

$(\cos\alpha L - \cosh\alpha L)(\sin\alpha L + \sinh\alpha L) = (\sin\alpha L - \sinh\alpha L)(\cos\alpha L + \cosh\alpha L)$

$\therefore \cancel{\cos\alpha L\sin\alpha L} + \cos\alpha L\sinh\alpha L - \cosh\alpha L\sin\alpha L - \cancel{\cosh\alpha L\sinh\alpha L}$

$= \cancel{\cos\alpha L\sin\alpha L} + \sin\alpha L\cosh\alpha L - \cos\alpha L\sinh\alpha L - \cancel{\sinh\alpha L\cosh\alpha L}$

$\therefore \cos\alpha L\sinh\alpha L = \sin\alpha L\cosh\alpha L$

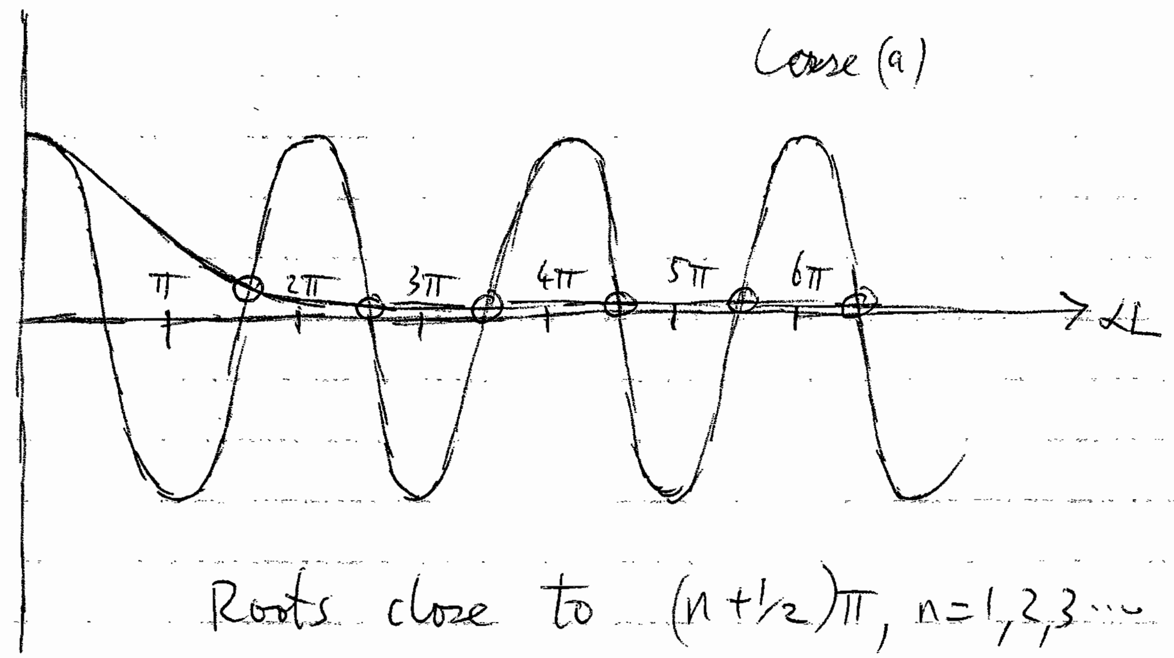
$\therefore \tan\alpha L = \tanh\alpha L$

(c)
To check for interlacing behaviour, rearrange as

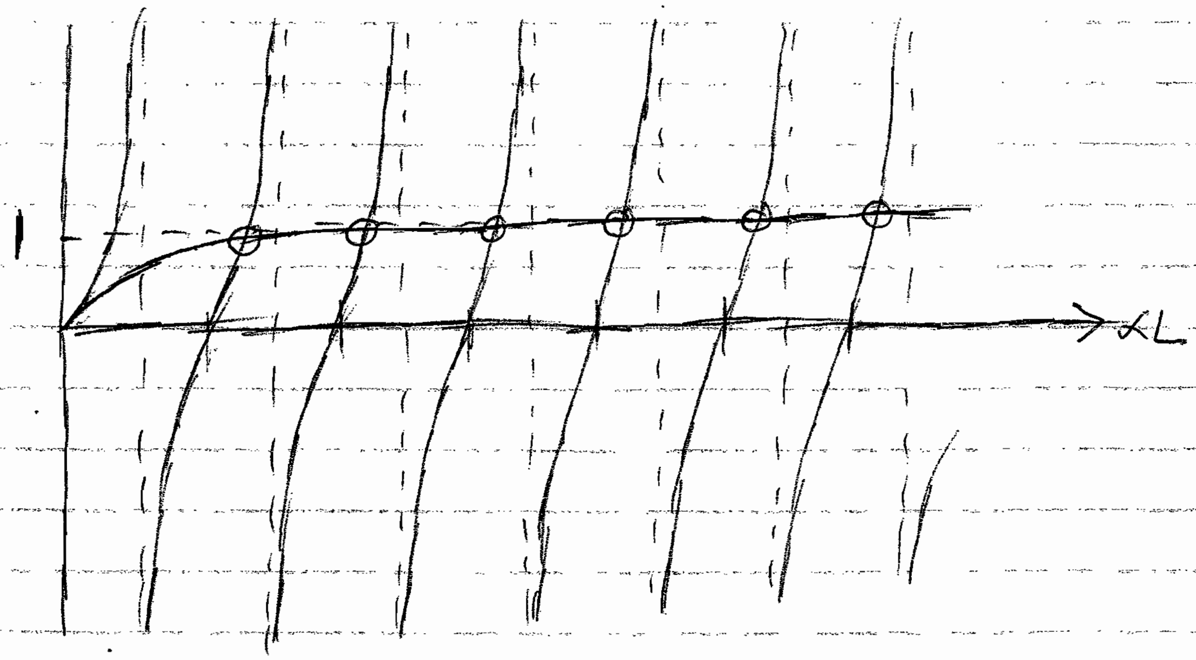
(a) $\cos\alpha L = \frac{1}{\cosh\alpha L}$, (b) $\tan\alpha L = \tanh\alpha L$

& plot on matching scales

1 cont:

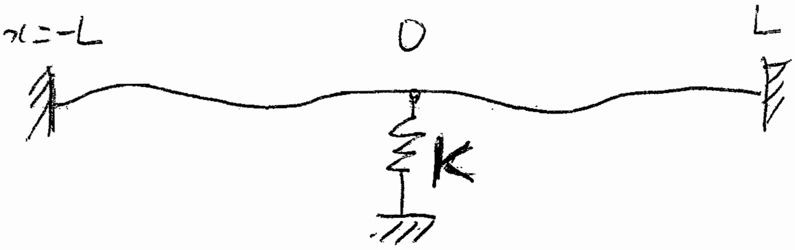



Roots close to $(n+1/2)\pi$, $n=1,2,3\cdots$




$\tanh \alpha L \to 1$ as $\alpha L$ increases, so roots are close to $(n+1/4)\pi$

So interlacing behaviour is clear

2(a) 


System is symmetric in line $x=0$. So behaviour must be unchanged when it is reflected in this line. Symmetric modes are identical, antisymmetric modes are the same except for a $180°$ phase reversal in the motion. Any other motion would change in a non-trivial way on reflection, so is not possible.

Antisymmetric modes have nodes at the spring, so are not influenced by it – they are the same as modes of a non-constrained string

Mode 1 : 

Mode 2 : 

For free string, $P\dfrac{\partial^2 w}{\partial x^2} = m\dfrac{\partial^2 w}{\partial t^2}$ (data sheet)

so if $w = u(x)e^{i\omega t}$, $u'' = -\dfrac{m\omega^2}{P}u = -k^2 u$ say

Then $u = A\cos kx + B\sin kx$ in general.

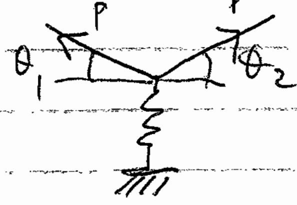
But antisymmetric, so $u = \sin kx$.

$u=0$ at $x=L$ $\rightarrow$ $\sin kL = 0$

$\therefore$ $k = \dfrac{n\pi}{L}$, so $\omega = k\sqrt{\dfrac{P}{m}} = \dfrac{n\pi}{L}\sqrt{\dfrac{P}{m}}$

(b) At $x=0$ must have force balance

$P\sin\theta_1 + P\sin\theta_2 = Kw$



2(b) cont. But $\theta_1, \theta_2$ small

So $\begin{cases} \sin\theta_1 \simeq \theta_1 \simeq -\left.\dfrac{\partial w}{\partial x}\right|_{x=0-} \\ \sin\theta_2 \simeq \theta_2 \simeq +\left.\dfrac{\partial w}{\partial x}\right|_{x=0+} \end{cases}$

$$\therefore \quad P\left[\frac{\partial w}{\partial x}\right]_{0-}^{0+} = Kw$$

For symmetric modes, let $u(x) = \sin k(x+L) \quad -L \le x \le 0$ with $k$ as in (a). This is a combination of $\sin kL$, $\cos kL$ which automatically satisfies $u = 0$ at $x = -L$.

Now use b.c. at $x = 0$:

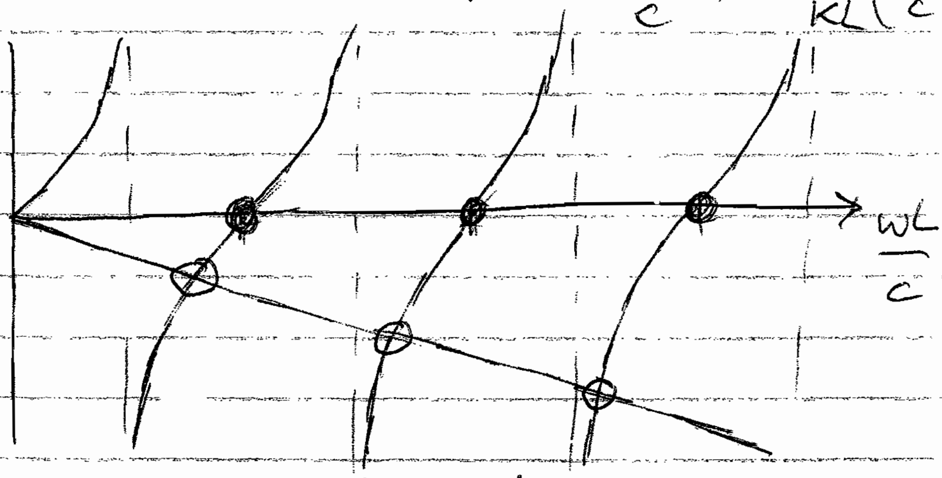
$$-2Pk\cos kL = K\sin kL$$

where $k = \omega/c$

$$\therefore \quad \tan\frac{\omega L}{c} = -\frac{2P\omega}{Kc}$$

(c) From (a), antisymmetric frequencies satisfy $\dfrac{\omega L}{c} = n\pi$.

From (b), draw plots of $\tan\dfrac{\omega L}{c}$, $-\dfrac{2P}{KL}\left(\dfrac{\omega L}{c}\right)$:



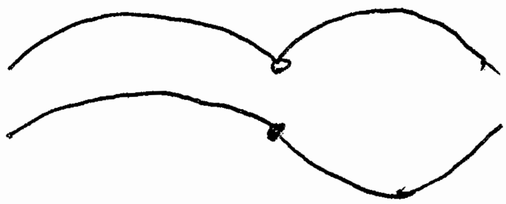

O = symmetric root

⊗ = antisymmetric solution

2(c) cont.

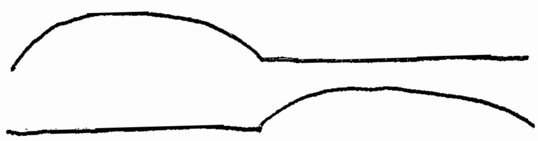
(i) $K \to 0$ means the straight line tends towards a vertical slope, and so the roots tend towards the asymptotes of the tan function, i.e. to $(n-\frac{1}{2})\pi$, $n = 1, 2, 3 \ldots$
These are the frequencies for an unconstrained string, as expected

(ii) $K \to \infty$ means the straight line becomes horizontal, so the roots tend to the same values as for the antisymmetric modes. The spring has become a rigid constraint so it divides the string into 2 separate portions. The two mods



and

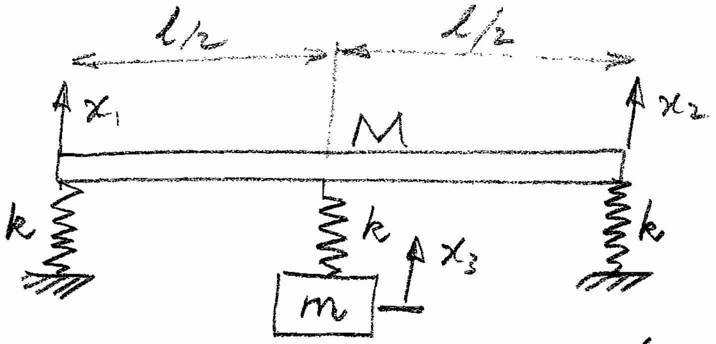
can then be combined via sum and difference to give



and

at the same frequency.

3. (a)




KE: $T = \frac{1}{2} m \dot{x}_3^2 + \frac{1}{2} M \left(\frac{\dot{x}_1 + \dot{x}_2}{2}\right)^2 + \frac{1}{2} I \left(\frac{\dot{x}_2 - \dot{x}_1}{l}\right)^2$

with $I = \frac{1}{12} M l^2$

So $T = \frac{1}{2} m \dot{x}_3^2 + \frac{1}{8} M \left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1 \dot{x}_2\right)$

$+ \frac{M}{24} \left(\dot{x}_2^2 + \dot{x}_1^2 - 2\dot{x}_1 \dot{x}_2\right)$

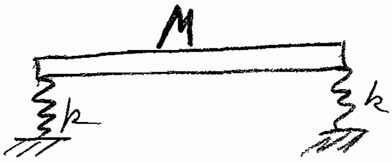
$= \frac{1}{2} m \dot{x}_3^2 + \frac{1}{6} M \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1 \dot{x}_2\right)$ //

PE: $V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k \left(\frac{x_1 + x_2}{2} - x_3\right)^2$

$= \frac{1}{2} k \left(x_1^2 + x_2^2 + \frac{x_1^2}{4} + \frac{x_2^2}{4} + \frac{x_1 x_2}{2} - (x_1 + x_2) x_3 + x_3^2\right)$

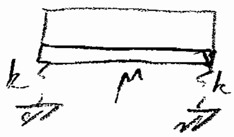
$= \frac{1}{2} k \left(\frac{5}{4} x_1^2 + \frac{5}{4} x_2^2 + \frac{x_1 x_2}{2} - x_1 x_3 - x_2 x_3 + x_3^2\right)$ //

(b)(i) $\frac{m}{M} \ll 1$: System is ≈



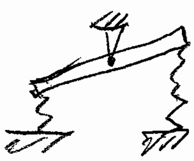

Mode | Frequency

$\begin{Bmatrix} 1 \\ 1 \\ 1+\varepsilon \end{Bmatrix}$



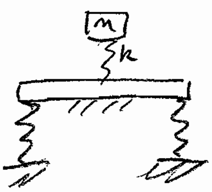

'bounce' of M with m "stuck" on $\omega_1 \approx \sqrt{\frac{2k}{M}}$ (approx)

$\begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix}$



pure 'pitch'
$\frac{1}{12} M l^2 \ddot{\theta} + \frac{k l^2}{2} \theta = 0 \Rightarrow \omega_2 = \sqrt{\frac{6k}{M}}$

$\begin{Bmatrix} -\varepsilon \\ -\varepsilon \\ 1 \end{Bmatrix}$

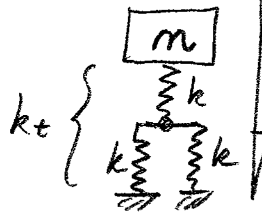



bounce of m
M is ≈ inertial

$\omega_3 = \sqrt{\frac{k}{m}}$ (approx)

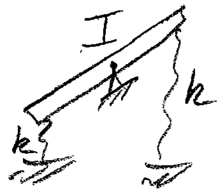
3(b)(ii) $\frac{m}{M} \gg 1$

Mode | frequency

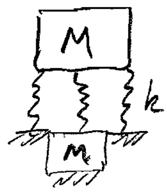
$\begin{Bmatrix} 1/3 \\ 1/3 \\ 1 \end{Bmatrix}$ 


$\frac{1}{k_T} = \frac{1}{k} + \frac{1}{2k} \Rightarrow k_t = \frac{2}{3}k \Rightarrow \omega_1 = \sqrt{\frac{2}{3}\frac{k}{m}}$ (approx)

bounce of $m$ on combined stiffness

$\begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$ 


pure pitch $\omega_3 = \sqrt{\frac{6k}{M}}$ (exact, unchanged)

$\begin{Bmatrix} 1 \\ 1 \\ -\varepsilon \end{Bmatrix}$ 


bounce of $M$ ($m$ is inertial) $\omega_2 = \sqrt{\frac{3k}{M}}$ (approx)

(c) for $M/m \ll 1$ $\quad (x_1\ x_2\ x_3)^T = (1\ 1\ \alpha)^T$

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\cancel{\tfrac{1}{2}}k\left(5/4 + 5/4 + 1/2 - \alpha - \alpha + \alpha^2\right)}{\cancel{\tfrac{1}{2}}\left[m\alpha^2 + \frac{M}{3}(1 + 1 + 1)\right]} = \frac{k(3 - 2\alpha + \alpha^2)}{m\alpha^2 + M}$$

Find exact frequencies by minimizing Rayleigh's quotient

$$\frac{d\omega^2}{d\alpha} = \frac{(m\alpha^2 + M)\cancel{k}(-2 + 2\alpha) - \cancel{k}(3 - 2\alpha + \alpha^2)(2m\alpha)}{(m\alpha^2 + M)^2}$$

$$\frac{d\omega^2}{d\alpha} = 0 \Rightarrow -2m\alpha^2 - 2M + \cancel{2m\alpha^3} + 2M\alpha - 6m\alpha + 4m\alpha^2 - \cancel{2m\alpha^3} = 0$$

$$\Rightarrow \cancel{2}m\alpha^2 + \alpha(\cancel{2}M - \cancel{6}^{3}m) - \cancel{2}M = 0$$

$$\alpha = \frac{-(M - 3m) \pm \sqrt{(M - 3m)^2 + 4mM}}{2m}$$

$$= \frac{3}{2} - \frac{M}{2m} \pm \sqrt{\left(\frac{M}{2m}\right)^2 - \frac{M}{2m} + \frac{9}{4}}$$

3 Cont

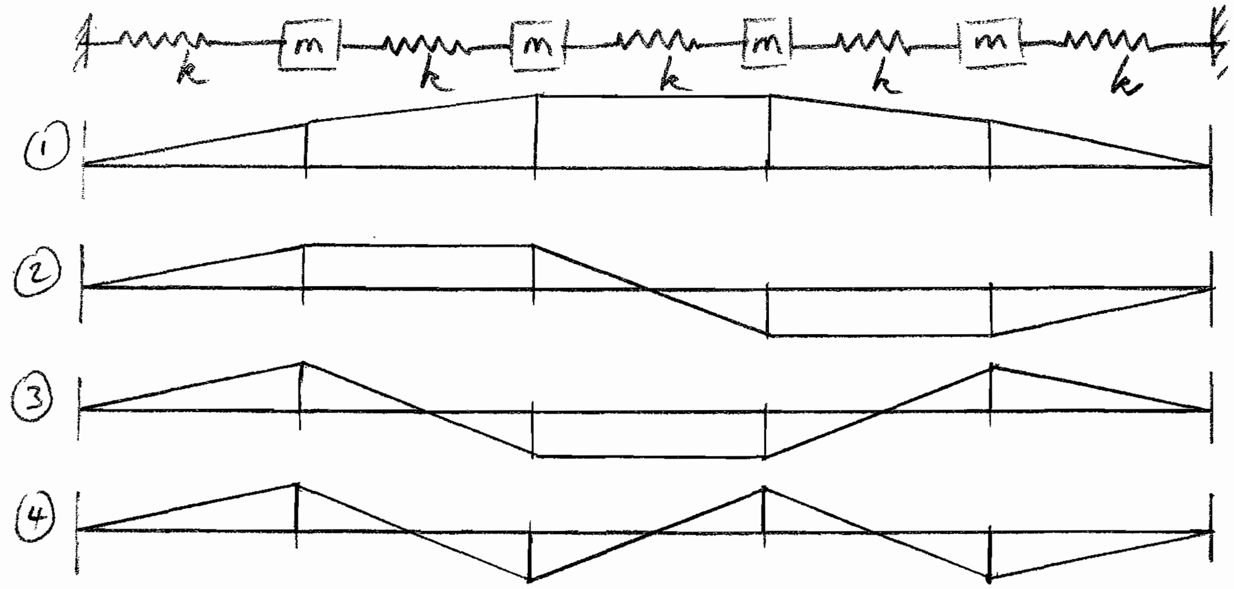
If $\frac{M}{m} \to 0$, $\alpha = \frac{3}{2} \pm \frac{3}{2} = \underline{3, 0}$

So modes are $\begin{Bmatrix} 1 \\ 1 \\ 3 \end{Bmatrix}$ & $\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ ✓

$\alpha = 0 \to \omega_2^2 = \frac{3k}{M}$ ✓

$\alpha = 3 \to \omega_1^2 = \frac{k(3-6+9)}{9m+M} = \frac{6k}{m(9+\cancel{\frac{M}{m}})} = \frac{2}{3}\frac{k}{m}$ ✓

4. (a)



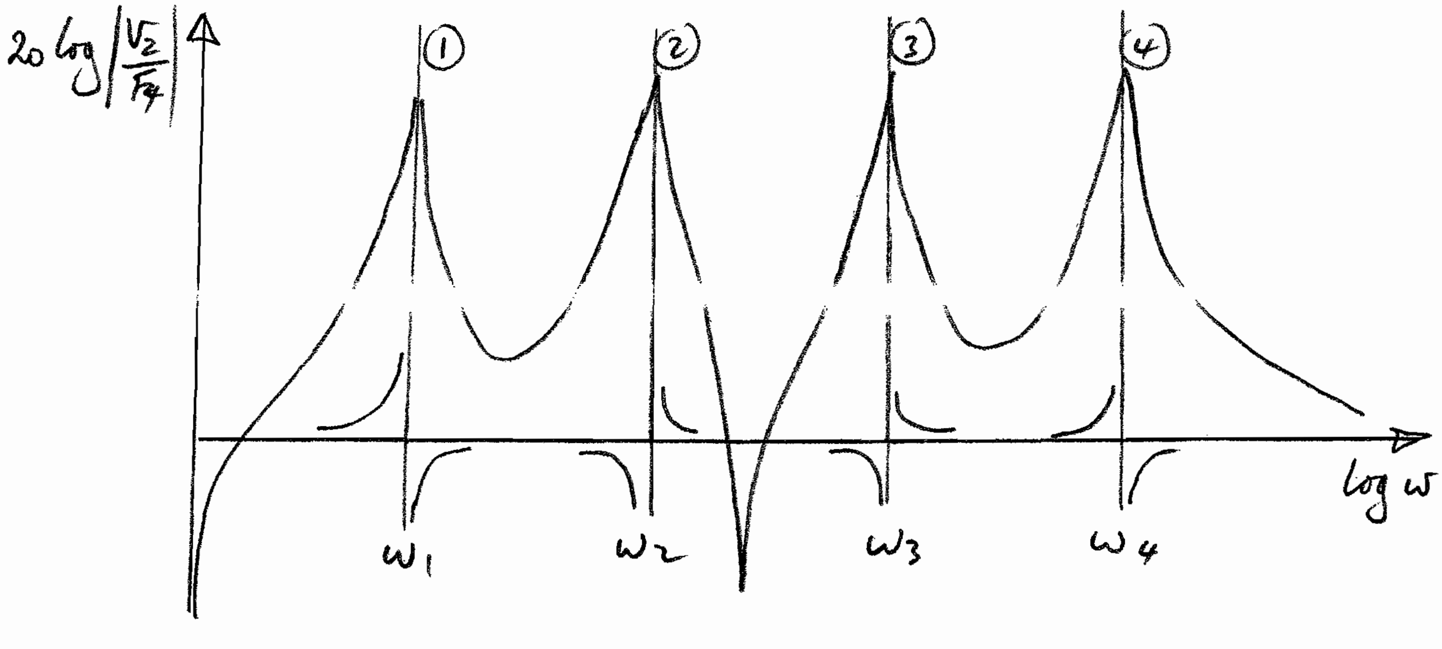

Modes 1 & 3 will be symmetric and a mode shape of the form $[1\ \alpha\ \alpha\ 1]^T$ will be suitable for both

Modes 2 & 4 will be anti-symmetric

(b) Products of modal amplitudes for points 2 & 4

1 +
2 −
3 −
4 +

So there is an anti-resonance between modes 2 & 3 and minima elsewhere



4 (c)

KE: $T = \frac{1}{2} m \left[ \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 \right]$

PE: $V = \frac{1}{2} k \left[ x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2 \right]$

So $[M] = m[I]$ & $[K] = k \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

(d) For mode $[1\ \alpha\ \alpha\ 1]^T$, Rayleigh's quotient is:

$$R = \omega^2 = \frac{V}{T^*} = \frac{\frac{1}{2} k \left[ 1 + (\alpha - 1)^2 + 0^2 + (\alpha - 1)^2 + 1 \right]}{\frac{1}{2} m \left[ 1 + \alpha^2 + \alpha^2 + 1 \right]}$$

$$= \frac{k}{m} \left( \frac{\alpha^2 - 2\alpha + 2}{\alpha^2 + 1} \right)$$

Minimise $R$:

$$\frac{dR}{d\alpha} = 0 \Rightarrow (\alpha^2 + 1)(2\alpha - 2) - (\alpha^2 - 2\alpha + 2)(2\alpha) = 0$$

$$\Rightarrow (\cancel{2\alpha^3} - 2\alpha^2 + 2\alpha - 2) - (\cancel{2\alpha^3} - 4\alpha^2 + 4\alpha) = 0$$

$$\Rightarrow \alpha^2 - \alpha - 1 = 0$$

$$\therefore \alpha_{1,2} = \frac{1}{2}(1 \pm \sqrt{5}) = -0.618,\ 1.618$$

Since $\alpha^2 = \alpha + 1$, $\omega^2 = \frac{k}{m} \frac{(3 - \alpha)}{(\alpha + 2)}$

For $\alpha = \frac{1}{2}(1 + \sqrt{5})$ (Mode 1 has $\alpha > 0$), $\omega_1^2 = \frac{k}{m} \left( \frac{3 - \frac{1}{2}(1 + \sqrt{5})}{2 + \frac{1}{2}(1 + \sqrt{5})} \right) = \frac{k}{m} \left( \frac{5 - \sqrt{5}}{5 + \sqrt{5}} \right)$ $0.38\, k/m$

For $\alpha = \frac{1}{2}(1 - \sqrt{5})$ (Mode 3 has $\alpha < 0$), $\omega_3^2 = \frac{k}{m} \left( \frac{5 + \sqrt{5}}{5 - \sqrt{5}} \right) = 2.618\, k/m$

Both modes are exact because the true mode shape is among the family of assumed modes, due to symmetry.