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3C7 Paper 2008

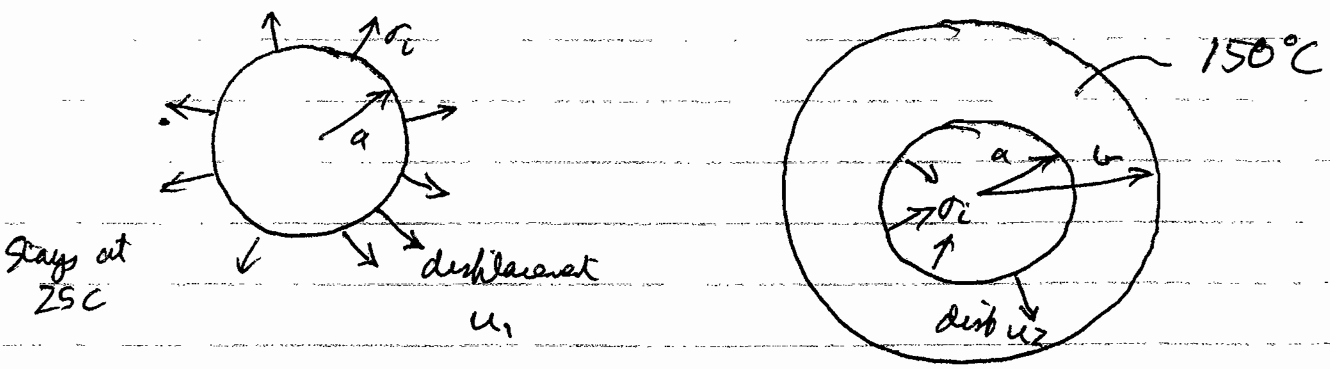
Solutions

1. (a) Plane stress occurs when there are no stresses on the surfaces of a thin sheet through the body. (15%)
If x, y are in the plane then $\sigma_{zz} = 0$ σ_{xz} & $\sigma_{yz} = 0$

(b) All dimensions increase with a strain of $\alpha(T - T_0)$ with no induced stresses (15%)

Surface displacement at the hole is $\underline{\underline{\alpha a (T - T_0)}}$

(c) Split into two:-



Basic method. Find displacements at interface in terms of σ_i and then use compatibility to find σ_i.

Inner disc $\sigma_{rr} = A - \dfrac{B}{r^2}$; $\sigma_{\theta\theta} = A + \dfrac{B}{r^2}$

$r \to 0$ stresses do not $\to \infty$ $\therefore B = 0$

$A = \sigma_i$

$= \sigma_{rr} = \sigma_{\theta\theta}$

$$\varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta} - \nu\sigma_{rr}}{E} = \frac{(1-\nu)\sigma_i}{E}$$

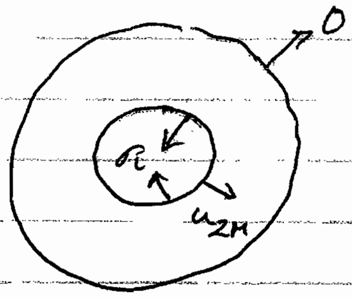
$$\varepsilon_{\theta\theta} = \frac{u}{r} \qquad \therefore \qquad u_1 = \frac{a(1-\nu)\sigma_i}{E}$$

Outer annulus.

Consider effect of heating and interference stress separately, then add and apply compatibility.

Thermal. $u_{2T} = a\alpha\Delta T$

Mechanical



$$\sigma_{rr} = A - \frac{B}{r^2}$$

$$\sigma_{\theta\theta} = A + \frac{B}{r^2}$$

$\sigma_{rr} = 0$ when $r = b$ $\qquad 0 = A - \dfrac{B}{b^2}$

$= \sigma_i \quad r = a$ $\qquad \sigma_i = A - \dfrac{B}{a^2}$

$$\therefore \sigma_i = \frac{B}{b^2} - \frac{B}{a^2} = \frac{(a^2-b^2)B}{a^2b^2}$$

$$\therefore B = \frac{a^2b^2}{(a^2-b^2)}\sigma_i \qquad A = \frac{a^2\sigma_i}{(a^2-b^2)}$$

$$E\varepsilon_{\theta\theta} = \sigma_{\theta\theta} - \nu\sigma_{rr} \quad \text{and} \quad \varepsilon_{\theta\theta} = \frac{u}{r}$$

$$\therefore u_{2M} = \frac{a}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})$$

$$\cancel{= a\varepsilon_{\theta i}}$$

$$(\sigma_{\theta\theta})_a = A + \frac{B}{a^2} = \frac{a^2\sigma_i}{(a^2-b^2)} + \frac{\cancel{a^2}b^2}{\cancel{a^2}(a^2-b^2)}\sigma_i$$

$$= \frac{(a^2+b^2)}{(a^2-b^2)}\sigma_i$$

$$(\sigma_{rr})_a = A - \frac{B}{a^2} = \frac{(a^2-b^2)}{(a^2-b^2)}\sigma_i = \sigma_i \quad \text{(as expected)}$$

$$\therefore u_{2M} = \frac{a}{E}\sigma_i\left(\frac{a^2+b^2}{a^2-b^2} - \nu\right)$$

$$\therefore u_1 = u_{2T} + u_{2M}$$

$$\frac{a}{E}(1-\nu)\sigma_i = a\alpha\Delta T + \frac{a}{E}\sigma_i\left(\frac{a^2+b^2}{a^2-b^2} - \nu\right)$$

$$E\alpha\Delta T = \sigma_i\left(\frac{(1-\nu)(a^2-b^2) - a^2 - b^2 + \nu a^2 - \nu b^2}{(a^2-b^2)}\right)$$

$$= \frac{\sigma_i}{(a^2-b^2)}\left(\cancel{a^2} - \cancel{\nu a^2} + \cancel{\nu b^2} - b^2 - \cancel{a^2} - b^2 + \cancel{\nu a^2} - \cancel{\nu b^2}\right)$$

$$= \frac{\sigma_i(-2b^2)}{(a^2-b^2)} \Rightarrow \sigma_i = \frac{E\alpha\Delta T\,(b^2-a^2)}{2b^2}$$

$$\text{or } \sigma_i = \frac{E\alpha\Delta T}{2}\left(1 - \frac{a^2}{b^2}\right)$$

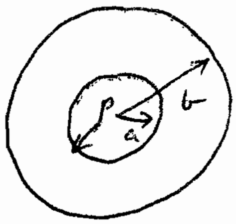
Put in numbers $\sigma_i = \frac{180.10^3 . 12.10^{-6} . 125}{2} \left(1 - \frac{(150)^2}{(250)^2}\right)$

$= \underline{\underline{86 \cdot 4 \text{ MPa}}}$

N.B. Poissons ratio not needed but given in question since if any candidate makes an error they won't be able to get any answer without it.

2. Take external pressure as 0, internal pressure as p.

(a)



Elastic. Apply Lame. Yield will occur on inner surface.

$$\sigma_{rr} = A - \frac{B}{r^2} \qquad \sigma_{\theta\theta} = A + \frac{B}{r^2}$$

$$\sigma_{rr} = -p \text{ @ } r = a \quad ; \quad \sigma_{rr} = 0 \text{ @ } r = b \quad \Rightarrow A = \frac{B}{b^2}$$

$$-p = \frac{B}{b^2} + \frac{B}{a^2} \quad \Rightarrow \frac{B(a^2+b^2)}{b^2a^2} = -p$$

$$B = \frac{-p\,b^2a^2}{(a^2+b^2)} \qquad A = \frac{-pa^2}{(a^2+b^2)}$$

$$\Rightarrow \sigma_{rr} = -p\left[\frac{a^2}{(a^2+b^2)} - \frac{b^2a^2}{(a^2+b^2).r^2}\right] = \frac{-p}{(1-m^2)}\left[1 - \frac{b^2}{r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{-p}{(1-m^2)}\left(1 + \frac{b^2}{r^2}\right)$$

Yielding will occur first on inner surface and ~~that~~ it must involve σ_{rr} & $\sigma_{\theta\theta}$ since σ_{zz} will be the intermediate stress (bonus if they discuss why!)

$$|\sigma_{\theta\theta} - \sigma_{rr}| = 2k = \frac{p}{(m^2-1)} \cdot \frac{2b^2}{r^2}$$

$$\Rightarrow \frac{p}{2k} = \frac{m^2-1}{2m^2}.$$

(b) Plastic everywhere.

$\therefore |\sigma_{\theta\theta} - \sigma_{rr}| = 2K$ throughout.

Equilibrium equation (data sheet) applies everywhere – not function of elastic/plastic.

$$\frac{\partial}{\partial r}(r\sigma_{rr}) + \cancel{\frac{\partial \sigma_{r\theta}}{\partial \theta}} - \sigma_{\theta\theta} = 0$$

0 assumption

$$\Rightarrow r.\frac{d\sigma_{rr}}{dr} + \sigma_{rr} - \sigma_{\theta\theta} = 0$$

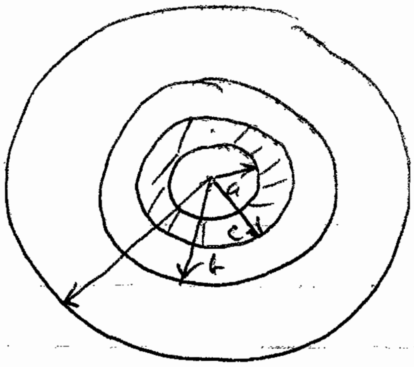
$$\therefore r.\frac{d\sigma_{rr}}{dr} = 2K$$

$$\Rightarrow \int_{-p}^{0} d\sigma_{rr} = \int_{a}^{b} \frac{2K}{r}$$

$$\Rightarrow \left[\sigma_{rr}\right]_{-p}^{0} = 2K \ln b/a$$

$$\Rightarrow \underline{\underline{\frac{p}{2K} = \ln m}}$$

(c) Need to consider three tubes



Inner plastic from a to c

Second elastic from c to b

Outer elastic from b to R.

Intermediate pressures. p on inside

$a = 100$

$b = 150$

$c = 135$

R is unknown

p_1 at c

p_2 at b

0 at R

For inner $\dfrac{p - p_1}{2k} = \ln\left(\dfrac{c}{a}\right)$ $\qquad \dfrac{240 - p_1}{420} = \ln \dfrac{135}{100}$

$\Rightarrow p_1 = 114$ MPa

Between c and b answer to (a) applies since yield just occurring at c

$$\therefore \frac{p_1 - p_2}{2K} = \frac{(b/c)^2 - 1}{2\,(b/c)^2} = \frac{(1{\cdot}11)^2 - 1}{2(1{\cdot}11)^2} \Rightarrow p_2 = 74 \text{ MPa}$$

From b out to D at point of yield at b.

$$\therefore \frac{p_2 - 0}{2k} = \frac{(R/b)^2 - 1}{2\,(R/b)^2} = \frac{74}{420} = 0{\cdot}176$$

$$\Rightarrow \left(\frac{R}{b}\right)^2 = 1{\cdot}54 \qquad \Rightarrow \frac{R}{b} = 1{\cdot}24 \qquad \Rightarrow R = 186 \text{ mm}$$

$$\Rightarrow D = \underline{\underline{372 \text{ mm.}}}$$

3 (a) $$\phi = \frac{\tau_0}{4}\left\{ xy - \frac{xy^2}{h} - \frac{xy^3}{h^2} + \frac{Ly^2}{h} + \frac{Ly^3}{h^2} \right\}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 0$$ everywhere (by inspection - no x^2 terms)

$$\frac{\partial \phi}{\partial y} = \frac{\tau_0}{4}\left\{ x - \frac{2xy}{h} - \frac{3xy^2}{h^2} + \frac{2Ly}{h} + \frac{3Ly^2}{h^2} \right\}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\tau_0}{4}\left\{ -\frac{2x}{h} - \frac{6xy}{h^2} + \frac{2L}{h} + \frac{6Ly}{h^2} \right\}$$

$$\sigma_{xy} = \frac{-\partial^2 \phi}{\partial x \partial y} = -\frac{\tau_0}{4}\left\{ 1 - \frac{2y}{h} - \frac{3y^2}{h^2} \right\}$$

(b) at $y = h$ $\sigma_{yy} = 0$ ✓✓

$$\sigma_{xy} = -\frac{\tau_0}{4}\left\{ 1 - 2 - 3 \right\} = +\tau_0$$ ✓✓

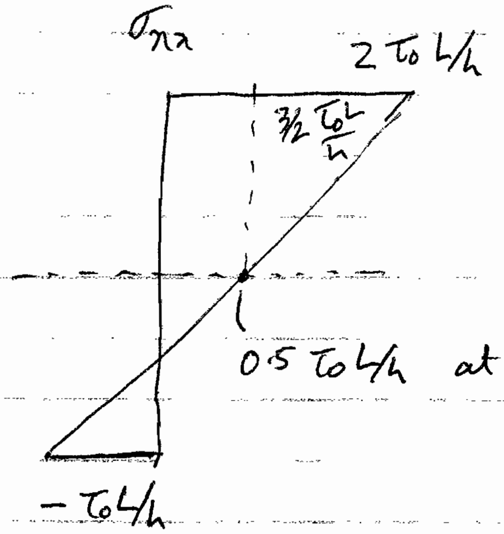
~~or~~

at $y = -h$ $\sigma_{yy} = 0$ ✓✓

$$\sigma_{xy} = -\frac{\tau_0}{4}\left\{ 1 + 2 - 3 \right\} = 0$$ ✓✓

(c) At $x=0$ $\quad \sigma_{xx} = \frac{\tau_0}{4}\left\{\frac{2L}{h} + \frac{6Ly}{h^2}\right\} = \frac{\tau_0 L}{h}\left(\frac{1}{2} + \frac{3}{2}\ldots\right.$

$$\sigma_{xy} = \frac{\tau_0}{4}\left\{-1 + \frac{2y}{h} + \frac{3y^2}{h^2}\right\}$$



$0.5\ \tau_0 L/h$ at $y=0$

∴ Horizontal Equilibrium should say

At root $\quad b\left(0.5\,\tau_0 \frac{L}{\cancel{h}}\right)\cancel{2h} = \underbrace{\tau_0 . L . b}$ applied by τ_0

<u>which is satisfied</u>

Moment equilibrium. Stress varies linearly

$$\therefore \text{Moment} = \frac{\sigma I}{y} = \frac{\cancel{3}}{\cancel{2}} . \frac{\tau_0 L}{h} . \frac{b(2h)^3}{\cancel{12}.h}$$

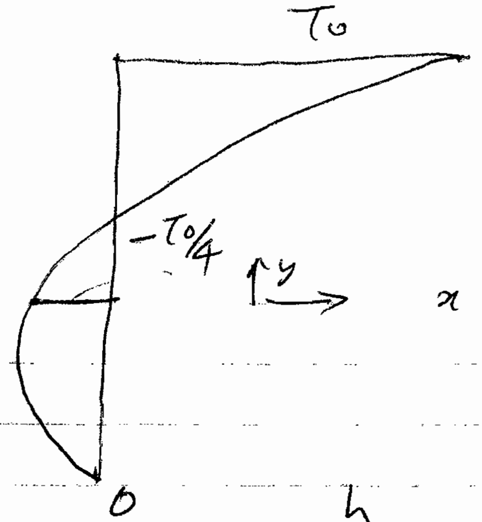
$$= \tau_0 L b h$$

same.

Applied moment $= \tau_0 . L . b . h$

∴ Moment equilibrium satisfied

Plot σ_{xy}



$$\text{Total vertical force} = \int_{-h}^{h} \sigma_{xy} \cdot b \cdot dy$$

$$= \frac{\tau_0 b}{4} \int_{-h}^{h} \left(-1 + \frac{2y}{h} + \frac{3y^2}{h^2} \right) dy$$

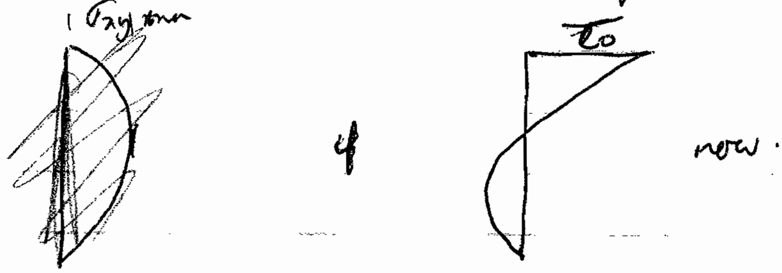
$$= \frac{\tau_0 b}{4} \left[-y + \frac{y^2}{h} + \frac{y^3}{h^2} \right]_{-h}^{+h}$$

$$= \frac{\tau_0 b}{4} \Big(\underbrace{(\cancel{-h} \cancel{+h} +h)}_{(\text{at } +h)} \underbrace{(\cancel{+h} \cancel{-h} + h)}_{(\text{at } -h)} \Big)$$

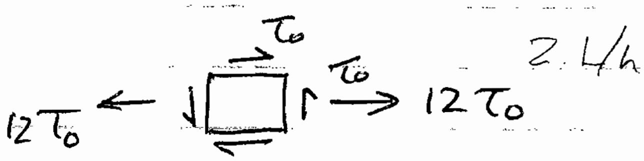
$= 0$ as expected.

(d) From (c) distribution of σ_{xx} is linear so this is the same as simple beam theory. ($\sigma_{xx} = 2\tau_0 L/h$
$= 12\tau_0$ here

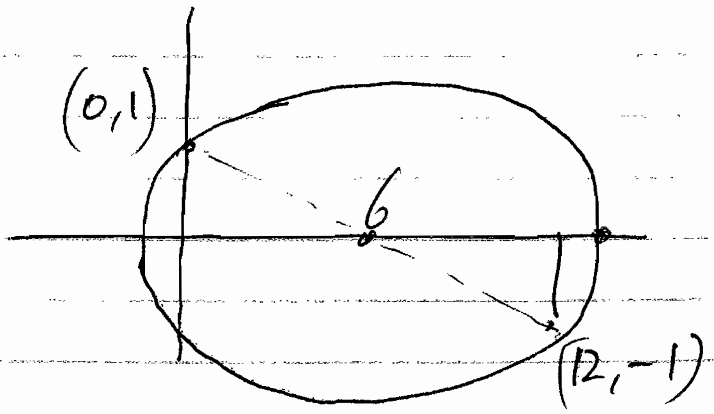
But simple distribution of shear would give



So



Mohr's circle

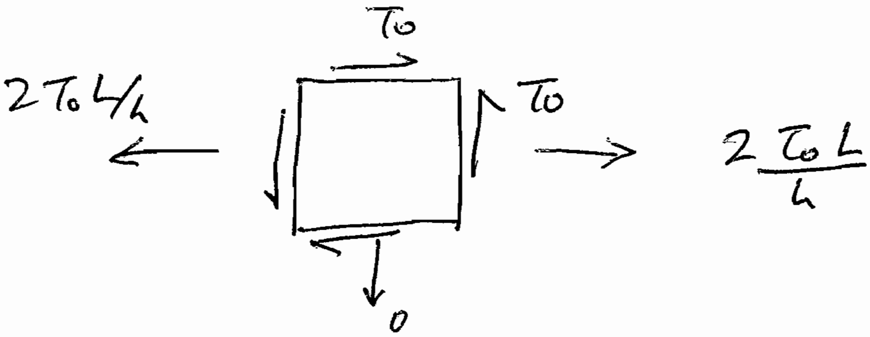


Radius $= \sqrt{37}$

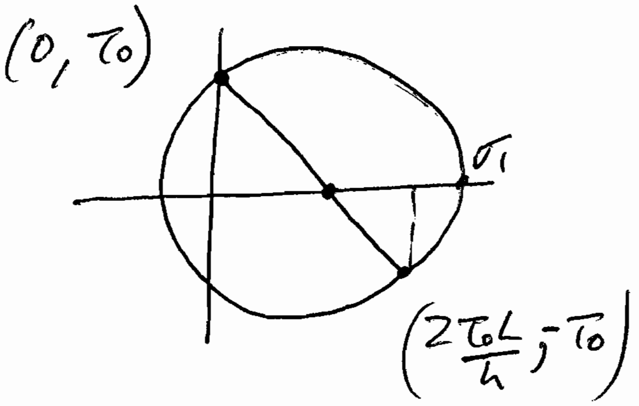
$\sigma_{max} = 6 + \sqrt{37} = 12.08\ \tau_0$ (< 1% difference)

cf $12\ \tau_0$

(d) The stress distribution at $x=0$, $y=h$ is



So Mohrs Circle



$\therefore$ Centre at $\frac{\tau_0 L}{h}$

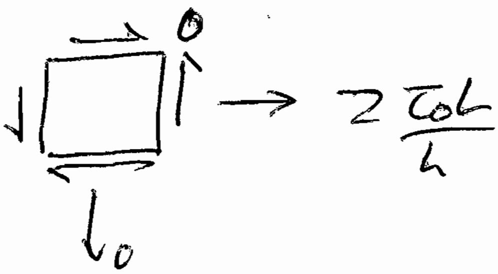
$$\text{Radius} = \sqrt{\tau_0^2 + \tau_0^2\left(\frac{L}{h}\right)^2}$$

$$= \tau_0\sqrt{1+\left(\frac{L}{h}\right)^2}$$

$\therefore$ Principal tensile stress will be

$$\sigma_1 = \frac{\tau_0 L}{h} + \tau_0\sqrt{1+\left(\frac{L}{h}\right)^2} = \frac{\tau_0 L}{h}\left(1 + \left(1+\left(\frac{h}{L}\right)^2\right)^{1/2}\right)$$

Simple beam theory gives

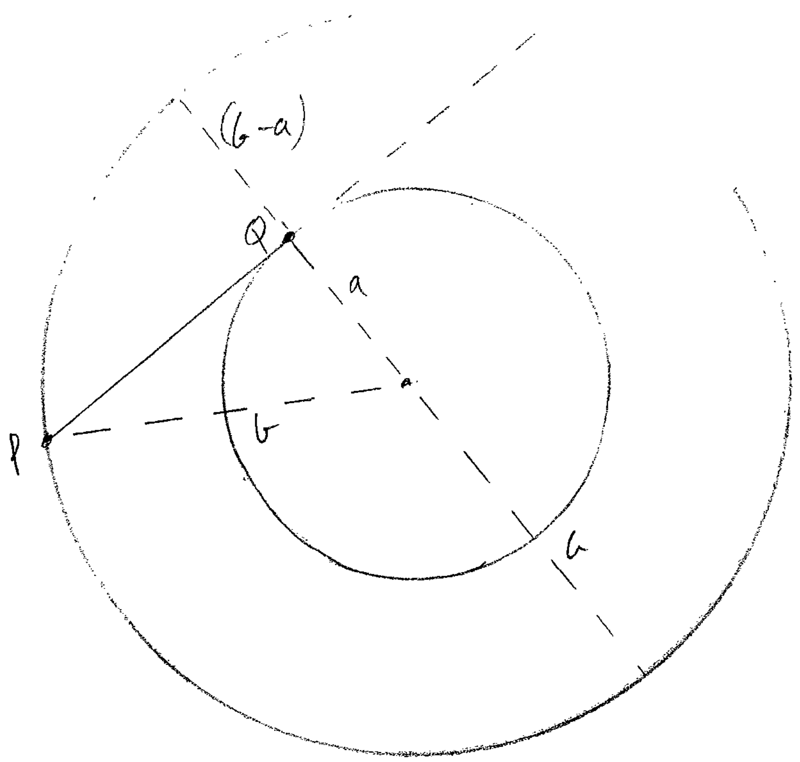


(No shear because no vertical load)

$$\sigma_{1s} = 2\frac{\tau_0 L}{h}$$

Bonus if they note that discrepancy gets larger as $h/L \to 1$.

(a)

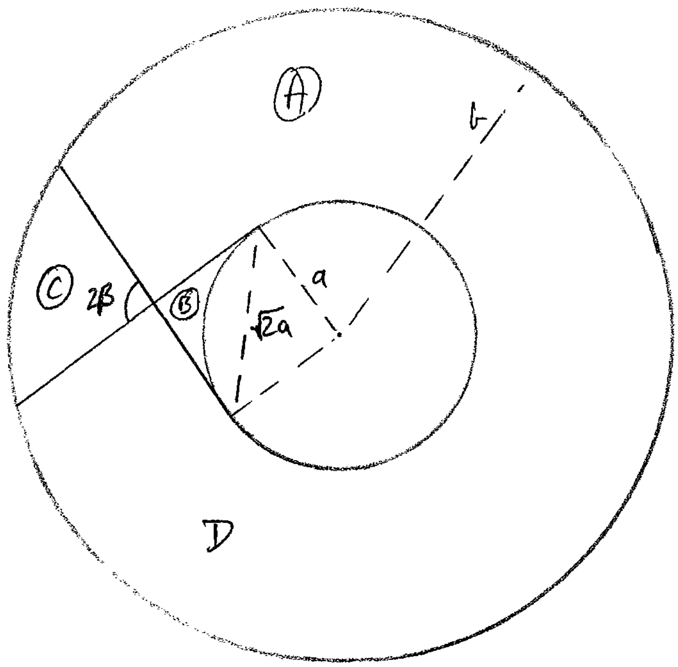


To find PQ

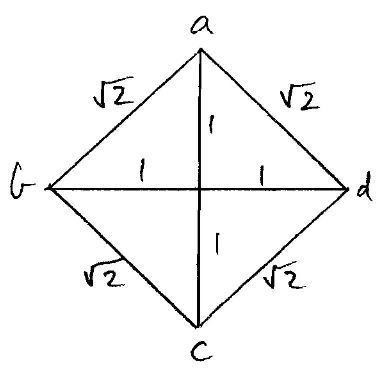
<u>either</u> Pythagoras $(PQ)^2 + a^2 = b^2$

$\Rightarrow PQ = \sqrt{b^2 - a^2} = \sqrt{(b-a)(b+a)}$

<u>or</u> Intersecting chords $PQ^2 = (b-a)(b+a)$



For $\beta = 45°$



Upper bound

$$4(p \cdot \sqrt{2}a \cdot 1) = k(l_{ab} \cdot v_{ab} + l_{ac} \cdot v_{ac}) \times 2 \times 2$$

$v_{ab} = v_{ac} = \sqrt{2}$ (from hodograph)

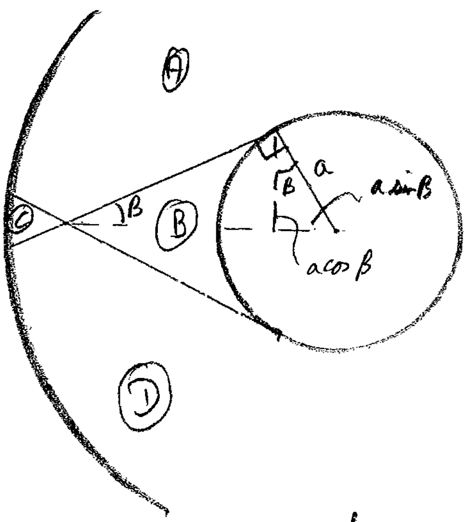
$l_{ab} = a$ $\quad l_{ab} + l_{ac} = PQ = \sqrt{b^2 - a^2}$

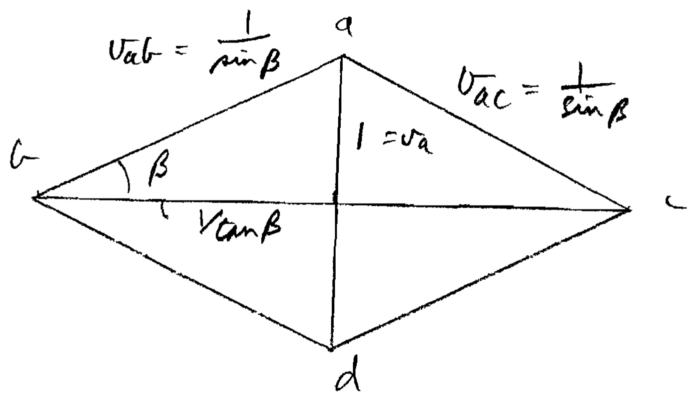
$$\therefore \cancel{4}\cancel{\sqrt{2}}\, p a = \cancel{4} k\left(\cancel{a\sqrt{2}} + \cancel{\sqrt{2}}\left(\sqrt{b^2-a^2} - \cancel{a}\right)\right)$$

$$p = k\left(\frac{\sqrt{b^2-a^2}}{a}\right) = k\left(\sqrt{(b/a)^2 - 1}\right)$$

If $b/a = 3$ $\quad \underline{p = 2\sqrt{2}k.}$

(b) If $\beta \neq 45°$





$$l_{ab} = \frac{a}{\tan\beta}$$

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Upper bound calculation

$$\cancel{2}\left(p \cdot \cancel{2}a \sin\beta \cdot 1 + k \cdot \cancel{2}a \cos\beta \cdot \frac{1}{\tan\beta}\right)$$

$$= \cancel{4} \cdot K \left(l_{ab} \cdot v_{ab} + l_{ac} \cdot v_{ac}\right)$$

But since $v_{ac} = v_{ab}$ then this $= \cancel{4} K v_{ab} (\underbrace{l_{ab} + l_{ac}}_{=PQ})$

$$\frac{\cos\beta}{\tan\beta} = \frac{\cos^2\beta}{\sin\beta}$$

$$\therefore \quad pa \left(\frac{\overbrace{\sin^2\beta + \cos^2\beta}^{1}}{\cancel{\sin\beta}}\right) = k \frac{1}{\cancel{\sin\beta}} \sqrt{(b^2 - a^2)}$$

$$p = k \sqrt{\left(\frac{b}{a}\right)^2 - 1}$$

(which is the same as before – possibly slightly surprising).