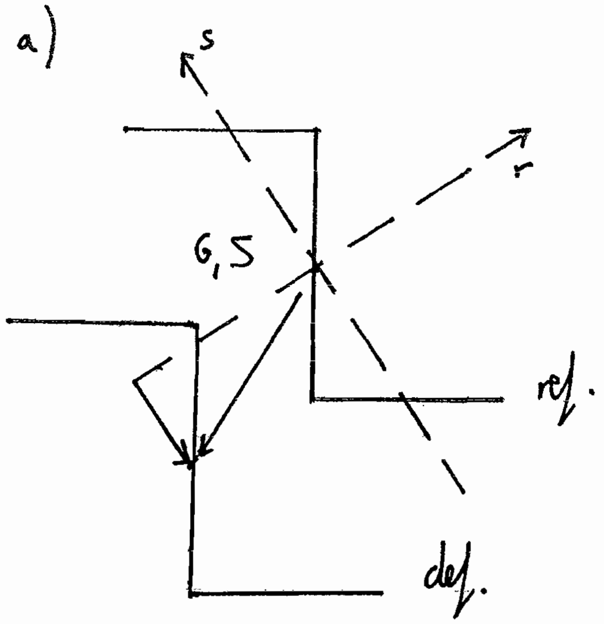
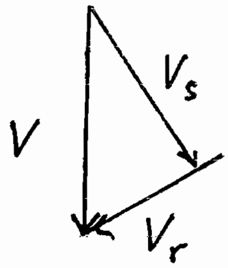
Dr F Cirak.

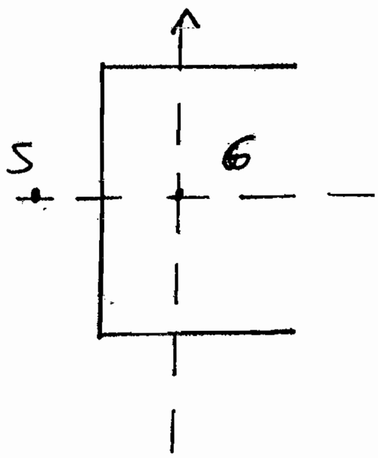
3D4 Structural Analysis and Stability - 2008

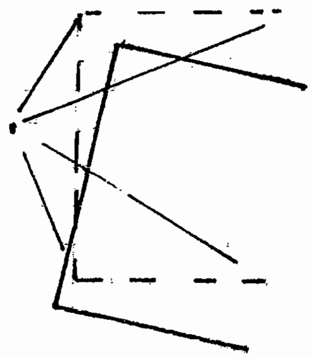
1 a)





Vertical + horizontal deflection
No rotation!





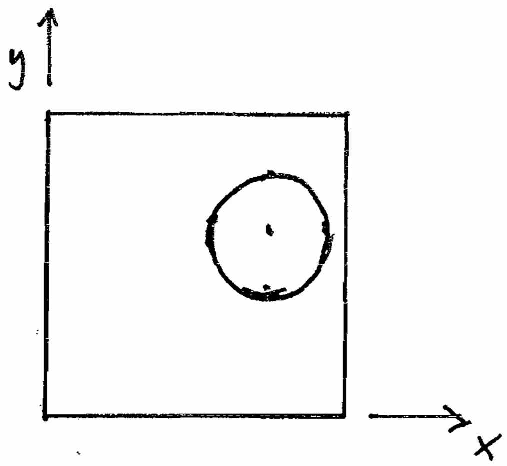
Vertical deflection + rotation
No horizontal deflection!

1 b)

Z-profile: No torsion; warping is not relevant

[-profile: Cross section warps; a warping constraint would make the beam stiffer
$\Rightarrow$ smaller deflections and rotations

1 b)



$A_c = \pi \cdot 0{,}75^2 = 1{,}767$ $\quad A = 16 - A_c = 14.23$

$$I_{sa} = \frac{4 \cdot 4^3}{12} = 21.33$$

$$I_c = \frac{0{,}75^4 \cdot \pi}{4} = 0{,}249$$

Center of gravity

$14.23\, x_s = 16 \cdot 2 - 1.767 \cdot 3 \Rightarrow x_s = 1.876$

$14{,}23\, y_s = 16 \cdot 2 - 1.767 \cdot 2{,}5 \Rightarrow y_s = 1{,}938$

Second moments of area

$$I_{xx} = 21{,}33 + 16 \cdot 0{,}062^2 - 0{,}249 - 1{,}767 \cdot 0{,}562^2$$
$$= 20{,}58$$

$$I_{yy} = 21{,}33 + 16 \cdot 0{,}124^2 - 0.249 - 1.767 \cdot 1{,}124^2$$
$$= 19{,}09$$

$$I_{xy} = 16 \cdot 0.124 \cdot 0.062 - 1.767 \cdot 0{,}562 \cdot 1{,}124$$
$$= -0.993$$

$$\tan 2\alpha = \frac{-0.993}{\left(\frac{20.58-19.09}{2}\right)} = -1.333$$

$$\Rightarrow \alpha = -26.56$$

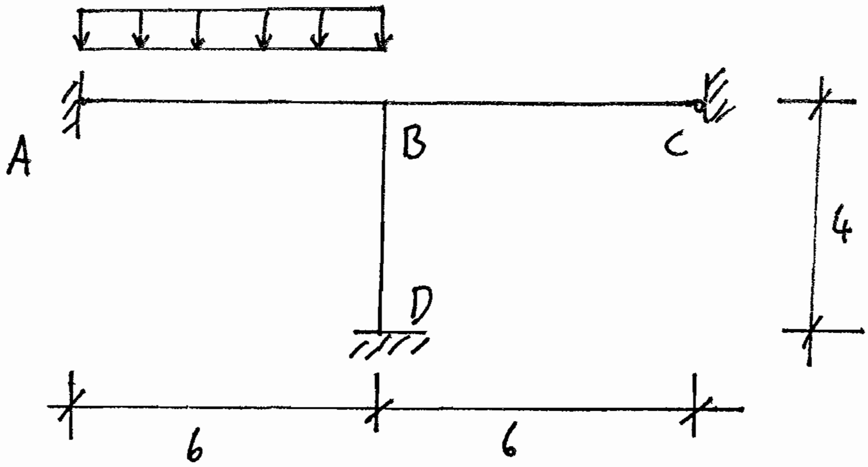
$$I_{\xi\xi} = \frac{20.58+19.09}{2} + \sqrt{\left(\frac{20.58-19.09}{2}\right)^2 + 0.993^2}$$

$$= 21.08$$

$$I_{\eta\eta} = \frac{20.58+19.09}{2} - \sqrt{\left(\frac{20.58-19.09}{2}\right)^2 + 0.993^2}$$

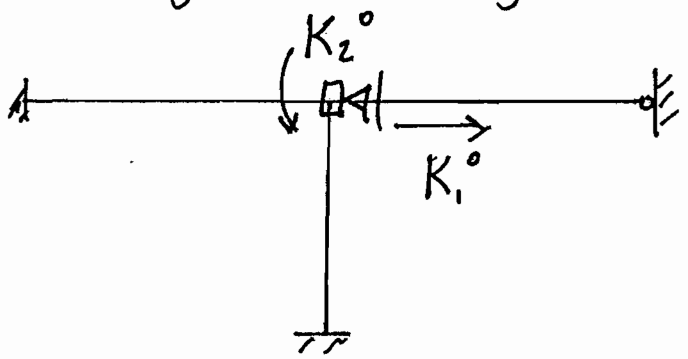
$$= 18.59$$

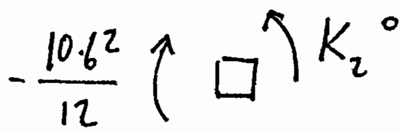
2)



$EI = 5 \cdot 10^4$
$EA = 6 \cdot 10^6$

Kinematically determined system

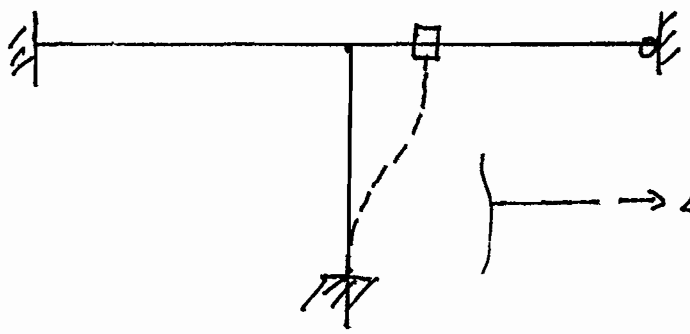


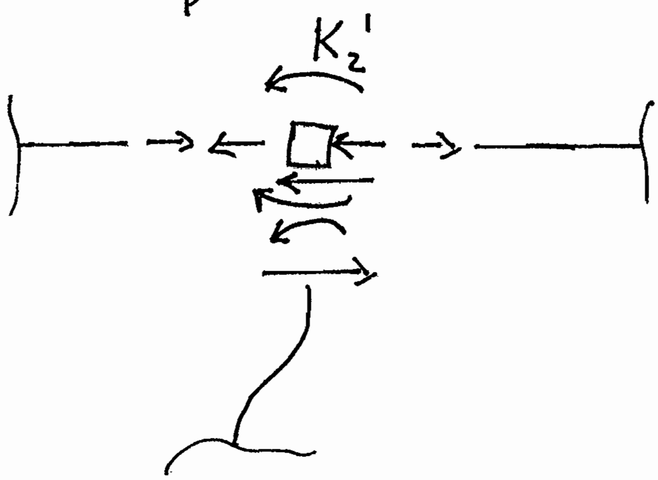


$$K_2^0 = \frac{-10 \cdot 6^2}{12} = -30$$

$$K_1^0 = 0$$

Unit displacement of node B

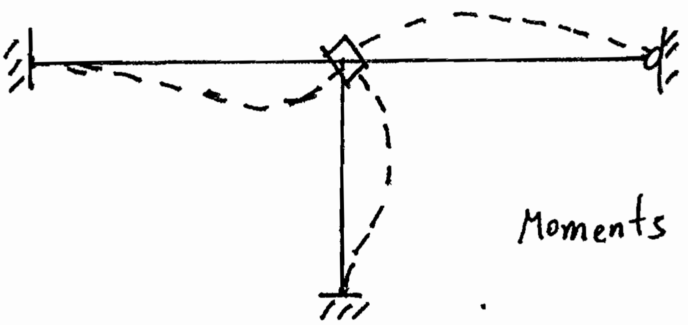




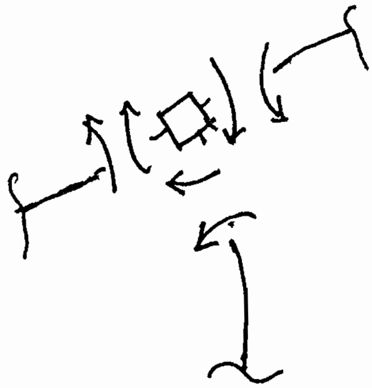
$$K_1^1 = 2 \cdot \frac{6 \cdot 10^6}{6} + \frac{12 \cdot 5 \cdot 10^4}{4^3}$$

$$= 2009375$$

Unit rotation of node B



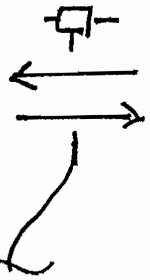
Moments at node B



$$K_2^2 = \frac{7 \cdot 5 \cdot 10^4}{6} + \frac{4 \cdot 5 \cdot 10^4}{4}$$

$$= \underline{108333}$$

Horizontal forces at node B



$$K_1^2 = \frac{6 \cdot 5 \cdot 10^4}{16} = 18750$$

Superposition

$$\begin{bmatrix} 0 \\ -30 \end{bmatrix} + \begin{bmatrix} 2009375 & 18750 \\ 18750 & 108333 \end{bmatrix} \begin{bmatrix} r^1 \\ r^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad r^2 = 2{,}774 \cdot 10^{-4}$$

$$r^1 = -2{,}588 \cdot 10^{-6}$$

Bending moments in the three sections adjacent to B

Beam AB

$$-30 + 0 \cdot r^1 + \frac{4 \cdot 5 \cdot 10^4}{6} r^2 = -20.753$$

Beam BD

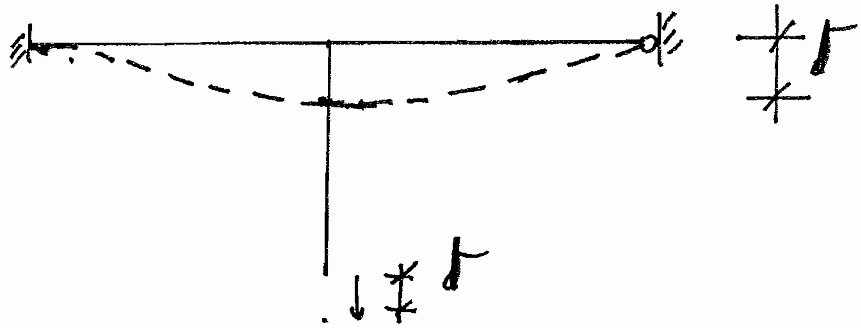
$$0 + \frac{6 \cdot 5 \cdot 10^4}{16} r^1 + \frac{4 \cdot 5 \cdot 10^4}{4} r^2 = 13,82$$

Beam BC

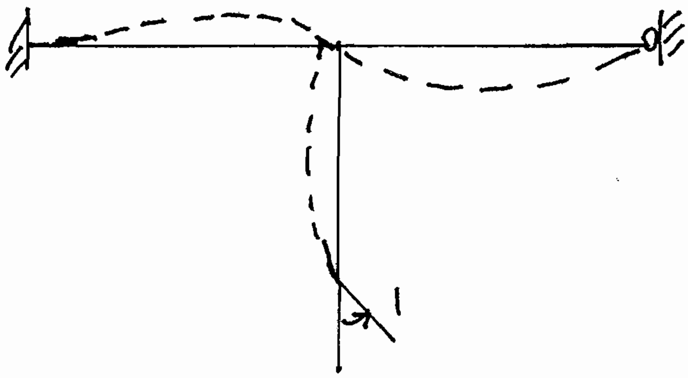
$$0 + 0 \cdot r^1 + \frac{3 \cdot 5 \cdot 10^4}{6} \cdot r^2 = 6,935$$

$$\sum M = -20,753 + 13,82 + 6.935 \approx 0 \checkmark$$

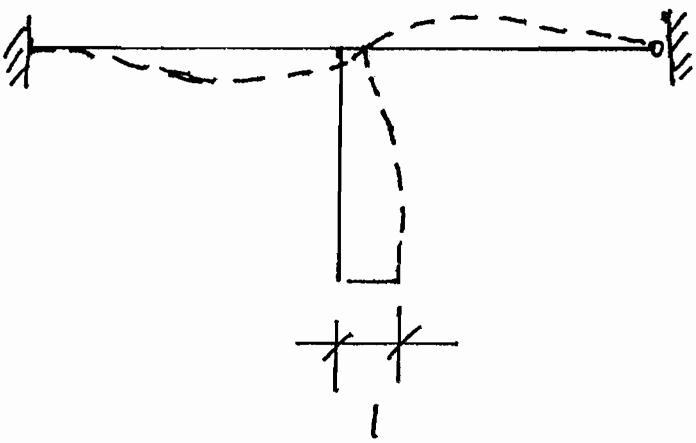
2c) Vertical Force



Moment



Horizontal Force



3D4 Structural Analysis and Stability 2008

3 (a) The eigenvalues of the stiffness matrix are the stiffnesses against deformation into the principal directions (the eigenvectors, or static mode shapes), and buckling corresponds to a eigenvalue going to zero (ie zero stiffness against deformation into that shape).

It is more general as based upon the Taylor Series expansion about <u>any</u> equilibrium of an elastic system of the total potential energy function

$$\Pi(\delta w) = \underbrace{\Pi(w)}_{\substack{= \text{ irrelevant} \\ \text{arbitrary} \\ \text{datum}}} + \underbrace{\frac{\partial \Pi}{\partial w} . \delta w}_{\substack{\text{zero} \\ \text{at} \\ \text{equilibrium}}} + \underbrace{\frac{1}{2} \delta w^T \frac{\partial^2 \Pi}{\partial w^2} \delta w}_{\substack{\text{governing term} \\ \text{(quadratic form)}}} + \dots$$

and $\frac{\partial^2 \Pi}{\partial w^2}$ is, the matrix of curvatures.

is the stiffness matrix.

The "eigenvalues are critical loads" view has limitations

→ needs clear separation into axial + lateral directions (and so can't deal with snap through)

→ needs axial deflections $\propto$ (lateral deflections)^2

→ is only valid at buckling, whereas alternative description is valid at <u>any</u> load.

→ etc.

3(b) i) $\Pi(w) = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 - P\left(\frac{dw}{dx}\right)^2 + kw^2 \; dx$

$$w = \sum_n a_n \sin\frac{n\pi x}{L}$$

$$\frac{dw}{dx} = \sum_n \frac{n\pi}{L} a_n \cos\frac{n\pi x}{L}$$

$$\frac{d^2w}{dx^2} = \sum_n -\left(\frac{n\pi}{L}\right)^2 a_n \sin\frac{n\pi x}{L}$$

The first term $\Pi(w)_{first} = \frac{1}{2}\int_0^L EI \sum_n\sum_m a_n a_m \left(\frac{n^2\pi^2}{L^2}\right)\left(\frac{m^2\pi^2}{L^2}\right) \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx$

but $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} L/2 & n=m \\ 0 & n\neq m\end{cases}$

$$\therefore \quad \Pi(w)_{first} = \frac{1}{2}EI\left(\frac{L}{2}\right)\sum_n a_n^2 \frac{n^4\pi^4}{L^4}$$

Similarly $\Pi(w)_{second} = -\frac{1}{2}P\left(\frac{L}{2}\right)\sum_n a_n^2 \frac{n^2\pi^2}{L^2}$

$$\Pi(w)_{third} = \frac{1}{2}k\left(\frac{L}{2}\right)\sum_n a_n^2$$

$\therefore \quad \Pi(w) = \frac{1}{2}a^T K_{TOT}\, a$ with K_{TOT} = diagonal

($\therefore$ Sine functions are eigenvectors)

Diagonal terms $K_{nn} = \frac{L}{2}\left[\frac{n^4\pi^4}{L^4}EI - P\frac{n^2\pi^2}{L^2} + k\right]$

Instability when

$K_{nn} = 0 \quad \rightarrow \quad P_{cr,n} = \frac{L^2}{n^2\pi^2}\left(n^4\pi^4\frac{EI}{L^4} + k\right)$

$$= \frac{n^2\pi^2 EI}{L^2} + \frac{kL^2}{n^2\pi^2}$$

3(b) cont'd.

For given n, minimum $P_{cr,n}$ from

$$\frac{dP_{cr,n}}{dL} = 0$$

$$= -\frac{2n^2\pi^2 EI}{L^3} + \frac{2kL}{n^2\pi^2}$$

$$\therefore \quad n^2\pi^2 EI = \frac{kL^4}{n^2\pi^2}$$

$$n^4\pi^4 \frac{EI}{k} = L^4$$

Length giving minimum $P_{cr,n}$ $\quad L_{cr,n} = \cancel{n\,\pi\sqrt[4]{\frac{EI}{k}}}\; n\pi\left(\frac{EI}{k}\right)^{1/4}$

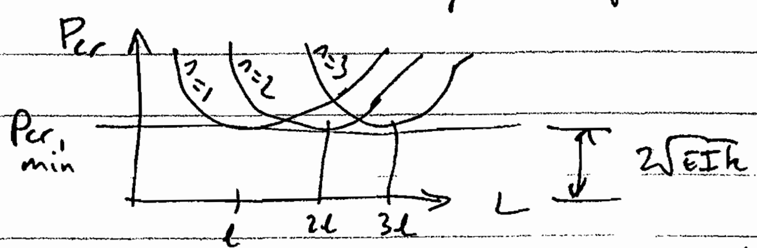
Min. value

$$P_{cr,n}(L_{cr,n}) = \frac{n^2\pi^2}{L_{cr,n}^2}EI + \frac{kL_{cr,n}^2}{n^2\pi^2}$$

$$= \frac{\cancel{n^2\pi^2}\, EI\sqrt{k}}{\cancel{n^2\pi^2}\sqrt{EI}} + k\frac{\cancel{n^2\pi^2}}{\cancel{n^2\pi^2}}\frac{\sqrt{EI}}{\sqrt{k}}$$

$$= 2\sqrt{EIk}$$

independent of n

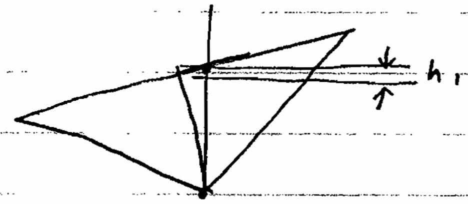


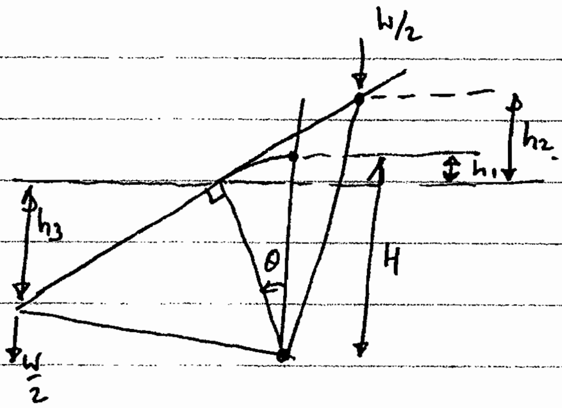
$$l = \pi\left(\frac{EI}{k}\right)^{1/4}$$

3D4

Q4. a) Total P.E. Π = Int. strain energy − W.D. by applied loads

$$= \tfrac{1}{2}G\theta^2 - \text{"}P.\Delta\text{"}$$





$$W.D. = \frac{W}{2}(h_3 + h_1) - \frac{W}{2}(h_2 - h_1)$$

$$h_2 = h_3$$

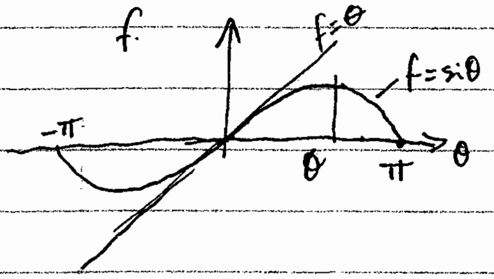
$$\therefore WD = \frac{W}{2}h_1 + \frac{W}{2}h_1 = Wh_1$$

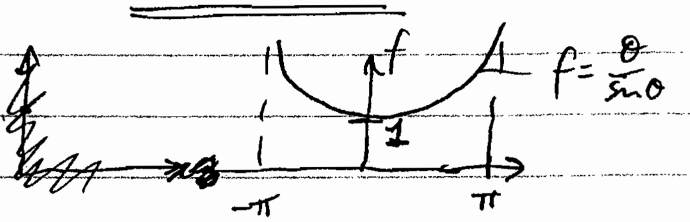
$$h_1 = H - H\cos\theta = H(1-\cos\theta)$$
$$WD = WH(1-\cos\theta)$$

$$\therefore \quad \underline{\Pi(\theta) = \tfrac{1}{2}G\theta^2 - WH(1-\cos\theta)} \qquad \text{Total pot. energy.}$$

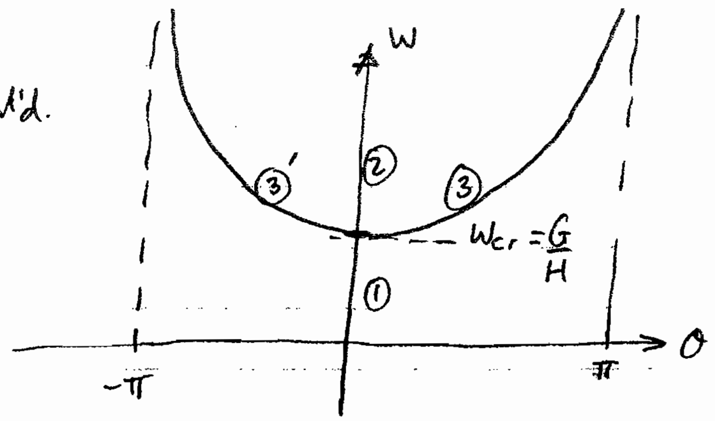
Equilib. from $\frac{\partial\Pi}{\partial\theta} = 0 \rightarrow G\theta - WH\sin\theta = 0$ for equilib.

Solutions; $\theta = 0$ or $\underline{\underline{W = \frac{G}{H}\frac{\theta}{\sin\theta}}}$





Q4 a) cont'd.



Stability from $\frac{\partial^2 \Pi}{\partial \theta^2} = G - WH\cos\theta$

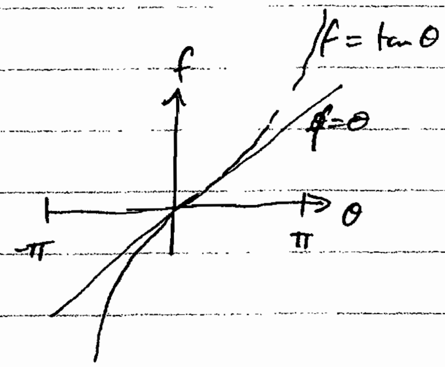
∴ Along $\theta = 0$ $\quad \frac{\partial^2 \Pi}{\partial \theta^2} = G - WH = \begin{cases} +\text{ve for } W < G/H & \text{stable (1)} \\ -\text{ve for } W > G/H. & \text{unstable (2)} \end{cases}$

Along $W = \frac{G}{H}\frac{\theta}{\sin\theta}$ $\quad \frac{\partial^2 \Pi}{\partial \theta^2} = G - WH\cos\theta$

(3) or (3)'

$$= G - \frac{G\theta\cos\theta}{\sin\theta}$$

$$= G\left(1 - \frac{\theta}{\tan\theta}\right)$$

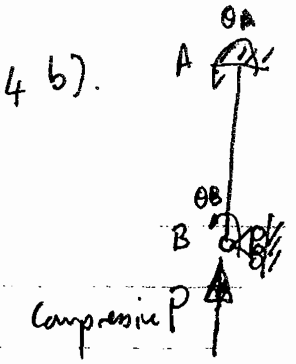


$\frac{\theta}{\tan\theta} < 1$ for $0 < |\theta| < \pi$

∴ $\frac{\partial^2 \Pi}{\partial \theta^2} = +\text{ve}$ ∴ stable on (3), (3)'.

Rotational instability when $W_{cr} = \frac{G}{H}$.

Q4 b).



$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = k \begin{bmatrix} s & sc \\ sc & s \end{bmatrix} \begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} \qquad k = \frac{EI}{L}$$

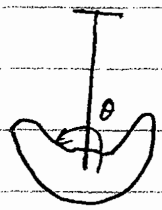
$$P_E = \frac{\pi^2 EI}{L^2}$$

$s = s(P/P_E)$ stiffness factor

$c = c(P/P_E)$ carryover factor.

$$\theta_A = 0 \rightarrow M_B = k s \theta_B$$

$\therefore$ G, rotational stiffness $= ks$



Now from earlier $W_{cr} = \dfrac{G}{H}$

$$\therefore W_{cr} = \frac{ks}{H} = \frac{EI}{L}\frac{1}{H} s = \frac{\pi^2 EI}{L^2}\frac{L}{H\pi^2} s$$

$$= \frac{1}{\pi^2}\frac{L}{H} P_E s$$

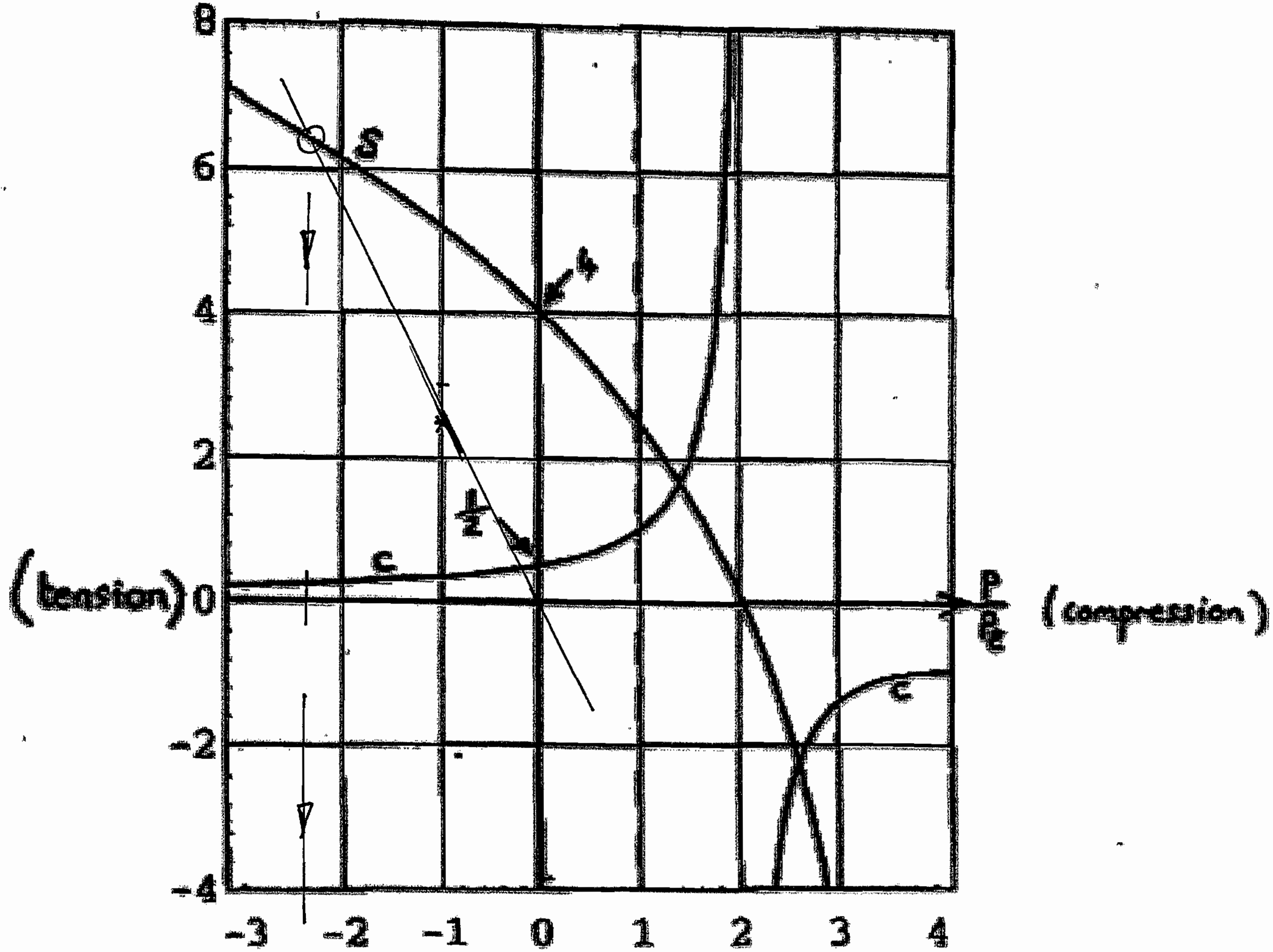
$$\therefore \left(\frac{\pi^2 H}{L}\right)\frac{W_{cr}}{P_E} = s$$

$\therefore$ say

$$\left.\begin{aligned} f_1 &= \frac{\pi^2 H}{L}\frac{W_{cr}}{P_E} \\ f_2 &= s \end{aligned}\right\} \text{ look for } f_1 = f_2$$

when $L = 4H$ $\quad f_1 \approx \dfrac{\pi^2}{4}\dfrac{W_{cr}}{P_E} \approx 2.5\dfrac{W_{cr}}{P_E}$

Now, $W = -P$ (W = tensile, P is compressive)



S
4
$\frac{1}{2}$
c
c
(tension)
(compression)
$\frac{P}{P_E}$
8
6
4
2
0
-2
-4
-3
-2
-1
0
1
2
3
4

Q4(b)

f_1, when $\frac{W_{cr}}{P_E} = 1$ $\quad f_1 = 2.5$ $\quad$ plot.

→ graph paper, (attached) $\quad f_1 = f_2$ when $\frac{P}{P_E} \approx -2.3$

$$\frac{W}{P_E} \approx 2.3$$

$$\therefore \quad W_{cr} = 2.3\,\pi^2 \frac{EI}{L^2}$$

Q4(c). Need to use "no-shear" stability functions s_1 and c_1