Dr D Symons

3D7 SOLUTIONS 2008

1) Equilibrium of node A:

$$\tfrac{1}{\sqrt{2}}t_I + \tfrac{1}{2}t_{II} - \tfrac{1}{2}t_{III} - \tfrac{1}{2}t_{IV} = P_{AX}$$

$$\tfrac{1}{\sqrt{2}}t_I + \tfrac{\sqrt{3}}{2}t_{II} + \tfrac{\sqrt{3}}{2}t_{III} - \tfrac{\sqrt{3}}{2}t_{IV} = P_{AY}$$

(a) $\therefore$

$$\underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & \sqrt{3}/2 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix}}_{\underset{\sim}{H}} \underbrace{\begin{bmatrix} t_I \\ t_{II} \\ t_{III} \\ t_{IV} \end{bmatrix}}_{\underset{\sim}{r}} = \underbrace{\begin{bmatrix} P_{AX} \\ P_{AY} \end{bmatrix}}_{\underset{\sim}{p}}$$

(b) For $\underset{\sim}{p} = \begin{bmatrix} W \\ 0 \end{bmatrix}$ write $[\underset{\sim}{H} \mid \underset{\sim}{p}]$ and eliminate

Working to 4d.p.

$$\left[\begin{array}{cccc|c} 0.7071 & 0.5 & -0.5 & -0.5 & W \\ 0.7071 & 0.8660 & 0.8660 & -0.8660 & 0 \end{array}\right]$$

Forward elimination

$$\rightarrow \left[\begin{array}{cccc|c} 0.7071 & 0.5 & -0.5 & -0.5 & W \\ 0 & 0.3660 & 1.3660 & -0.3660 & -W \end{array}\right]$$

Back elimination

$$\rightarrow \left[\begin{array}{cccc|c} 0.7071 & 0 & -2.3661 & 0 & 2.3661\,W \\ 0 & 0.3660 & 1.3660 & -0.3660 & -W \end{array}\right]$$

1) (b) (cont.)

Re-normalise $\rightarrow$
$$\left[\begin{array}{cccc|c} 1 & 0 & -3.3462 & 0 & 3.3462\,W \\ 0 & 1 & \underbrace{3.7322 \quad -1}_{\text{columns without pivots}} & & -2.7322\,W \end{array}\right]$$

Read off general solution:

$$\underset{\sim}{r} = \underset{\sim}{r_0} + \underset{\sim}{S}\,\underset{\sim}{x} = \begin{bmatrix} 3.3462\,W \\ -2.7322\,W \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3.3462 & 0 \\ 3.7322 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ {}_2 \end{bmatrix}$$

(c) Now solve $\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{S}\, \underset{\sim}{x} = -\underset{\sim}{S}^T(\underset{\sim}{F}\,\underset{\sim}{r_0} + \underset{\sim}{e_0})$

Because initially unstressed $\underset{\sim}{e_0} = 0$

Flexibility matrix $\underset{\sim}{F} = \dfrac{L}{AE}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 2/\sqrt{3} & 0 & 0 \\ 0 & 0 & 2/\sqrt{3} & 0 \\ 0 & 0 & 0 & 2/\sqrt{3} \end{bmatrix}$

$$\underset{\sim}{S}^T = \begin{bmatrix} -3.3462 & 3.7322 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$$

$$\underset{\sim}{S}^T \underset{\sim}{F} = \begin{bmatrix} -4.7322 & 4.3096 & -1.1547 & 0 \\ 0 & -1.1547 & 0 & -1.1547 \end{bmatrix} \frac{L}{AE}$$

$$\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{S} = \begin{bmatrix} 33.0738 & -4.3096 \\ -4.3096 & 2.3094 \end{bmatrix} \frac{L}{AE}$$

$$\underset{\sim}{S}^T \underset{\sim}{F}\, \underset{\sim}{r_0} = \begin{bmatrix} -27.6095\,W \\ 3.1549\,W \end{bmatrix} \frac{L}{AE}$$

1) (c) (cont.)

Hence system of compatibility equations is:

$$\begin{bmatrix} 33.0738 & -4.3096 \\ -4.3096 & 2.3094 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -27.6095\,W \\ 3.1549\,W \end{bmatrix}$$

$$\therefore \begin{bmatrix} 33.0738 & -4.3096 \\ 0 & 1.7478 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} +27.6095\,W \\ +0.4427\,W \end{bmatrix}$$

hence $x_2 = +0.2533\,W$ and $x_1 = +0.8678\,W$

so

$$\underset{\sim}{r} = \begin{bmatrix} 3.3462\,W \\ -2.7322\,W \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3.3462 & 0 \\ 3.7322 & -1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} +0.8678\,W \\ +0.2533\,W \end{bmatrix}$$

$$= \begin{bmatrix} 0.4424\,W \\ 0.2533\,W \\ -0.8678\,W \\ -0.2533\,W \end{bmatrix}$$

2) (a) Equivalent nodal loads may be calculated from

$$P_{jx} = t\int_A f_x n_j \, dA \quad \text{and} \quad P_{jY} = t\int_A f_Y n_j \, dA$$

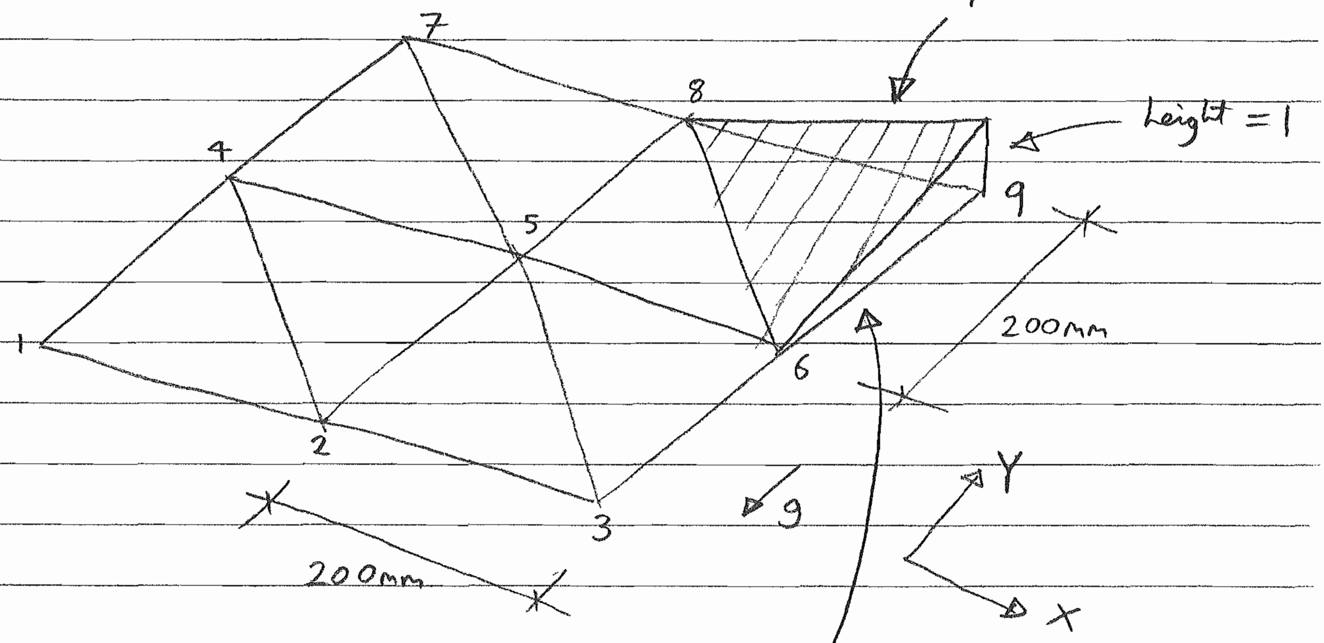
where n_j is the shape function for node j

f_x and f_Y are the components of distributed load

and A is the area of the whole mesh

Here $f_x = 0 \implies P_{9x} = 0$

and $f_Y = -\rho g$



$$P_{9Y} = t\int_A -\rho g \, n_9 \, dA$$

Volume of pyramid

$$= -\rho g t \int_A n_9 \, dA$$

$$= -3000 \times 10 \times 10^{-3} \times \frac{1}{3} \times \frac{1}{2} \times 0.2^2$$

$$= -0.2\,N$$

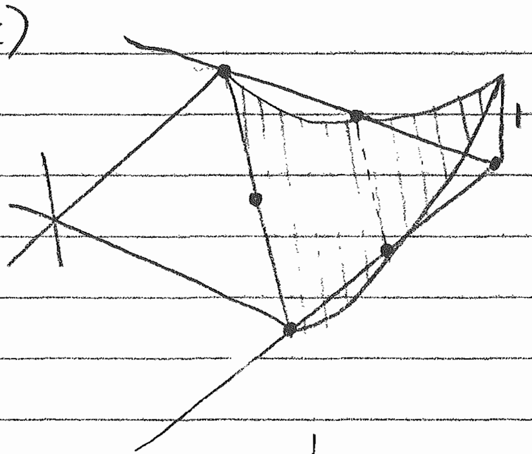
2) (b) Because $f_x = 0$ $P_{jX} = 0$ for all nodes ($j = 1$ to 9)

P_{jY} for each node is determined as follows

Node j	No. of elements @ node	Boundary conds	P_{jY}
1	1	$u_{1X} = u_{1Y} = 0$	0
2	3		$-0.6N$
3	2		$-0.4N$
4	3	$u_{4X} = u_{4Y} = 0$	0
5	6		$-1.2N$
6	3		$-0.6N$
7	2		$-0.4N$
8	3		$-0.6N$
9	1		$-0.2N$

$P_{1Y} = P_{4Y} = 0$ because these nodes (1 & 4) are constrained

(c)



$n_9 = 0$ at the mid-side nodes and therefore the integral $t\int_A f_Y n_9 \, dA$ is reduced

hence P_{9Y} is reduced

whereas $P_{9X} = 0$ still and is unchanged

2) (d)

The change to 6 noded, linear strain, triangular elements will give a much improved model of the displacement field (now quadratic rather than linear) and strain/stress field (now linear rather than constant within each element).

However the change significantly increases the computational cost, due to the increase in degrees of freedom of the mesh and because the elements are more complex.

3. (a) Applying a weight function v to the governing equation and integrate between x= 0 and x=3.

$$v\left[\frac{d}{dx}(Ak\frac{dT}{dx})\right]=0$$

$$\int_0^3 v\left[\frac{d}{dx}(Ak\frac{dT}{dx})\right]dx=0$$

Integrate by parts to derive the weak form.

$$\left[vAk\frac{dT}{dx}\right]_0^3-\int_0^3\frac{dv}{dx}Ak\frac{dT}{dx}dx=0$$

Heat flux is defined as $q=-k(dT/dx)$

$$\int_0^3\frac{dv}{dx}Ak\frac{dT}{dx}dx=(vAq)_{x=0}-(vAq)_{x=3}$$

$$\int_0^3\frac{dv}{dx}Ak\frac{dT}{dx}dx=0.3(v)_{x=0}q_0-(v)_{x=3}0.05\cdot 15$$

$$\int_0^3\frac{dv}{dx}Ak\frac{dT}{dx}dx=0.3(v)_{x=0}q_0-0.75(v)_{x=3}$$

(b) Substituting the shape functions given and noting that a is not a function of x,

$$\left(\int_0^3 \mathbf{Bc}Ak\mathbf{B}dx\right)\mathbf{a}=0.3(\mathbf{Nc})_{x=0}q_0-0.75(\mathbf{Nc})_{x=3}$$

Using vector product manipulation, $\mathbf{Bc}=\mathbf{c}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}$, $\mathbf{Nc}=\mathbf{c}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}$

$$\left(\int_0^3 \mathbf{c}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}Ak\mathbf{B}dx\right)\mathbf{a}=0.3\left(\mathbf{c}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}\right)_{x=0}q_0-0.75\left(\mathbf{c}^{\mathbf{T}}\mathbf{N}^{\mathbf{T}}\right)_{x=3}$$

Since $\mathbf{c}$ is not a function x,

$$\mathbf{c}^{\mathbf{T}}\left(\int_0^3 \mathbf{B}^{\mathbf{T}}Ak\mathbf{B}dx\right)\mathbf{a}=\mathbf{c}^{\mathbf{T}}[0.3\left(\mathbf{N}^{\mathbf{T}}\right)_{x=0}q_0-0.75\left(\mathbf{N}^{\mathbf{T}}\right)_{x=3}]$$

Cancelling $\mathbf{c}^{\mathbf{T}}$, the following finite element formulation can be obtained.

$$\left(\int_0^3 \mathbf{B}^{\mathbf{T}}Ak\mathbf{B}dx\right)\mathbf{a}=0.3\left(\mathbf{N}^{\mathbf{T}}\right)_{x=0}q_0-0.75\left(\mathbf{N}^{\mathbf{T}}\right)_{x=3}$$

(c) B = dN/dx = [-1, 1] for all elements

Element 1

$$\mathbf{K}_1^e = \int_0^1 \mathbf{B}^{eT} Ak \mathbf{B}^e dx = \int_0^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} Ak \begin{bmatrix} -1 & 1 \end{bmatrix} dx = 0.3 \cdot 5 \int_0^1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}$$

Element 2

$$\mathbf{K}_2^e = \int_1^2 \mathbf{B}^{eT} Ak \mathbf{B}^e dx = \int_1^2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} Ak \begin{bmatrix} -1 & 1 \end{bmatrix} dx = 0.2 \cdot 2 \int_1^2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix}$$

Element 3

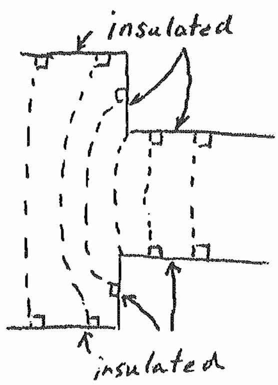
$$\mathbf{K}_3^e = \int_2^3 \mathbf{B}^{eT} Ak \mathbf{B}^e dx = \int_2^3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} Ak \begin{bmatrix} -1 & 1 \end{bmatrix} dx = 0.05 \cdot 3 \int_2^3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx = \begin{bmatrix} 0.15 & -0.15 \\ -0.15 & 0.15 \end{bmatrix}$$

(iv)

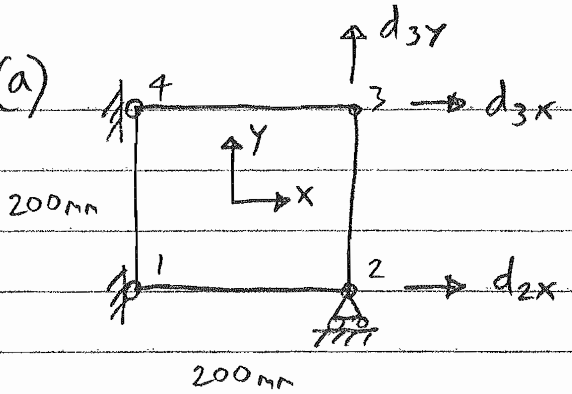
$$\begin{bmatrix} 1.5 & -1.5 & 0 & 0 \\ -1.5 & 1.9 & -0.4 & 0 \\ 0 & -0.4 & 0.55 & -0.15 \\ 0 & 0 & -0.15 & 0.15 \end{bmatrix} \begin{bmatrix} 50 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0.3q_0 \\ 0 \\ 0 \\ -0.75 \end{bmatrix}$$

Solve for q_0, T_2, T_3 and T_4.

(v)



4) (a)



$\nu = 0$

$E = 100\ kN/mm^2$

$t = 1\,mm$

Scaling from the data sheet we can find the shape functions of the unconstrained nodes:

$$n_2 = \frac{1}{4}(1 + 10X)(1 - 10Y) = \frac{1}{4}(1 + 10X - 10Y - 100XY)$$

$$n_3 = \frac{1}{4}(1 + 10X)(1 + 10Y) = \frac{1}{4}(1 + 10X + 10Y + 100XY)$$

Check: $n_2(-0.1, -0.1) = 0$ ✓ $\quad n_2(0.1, -0.1) = 1$ ✓

$n_2(0.1, 0.1) = 0$ ✓ $\quad n_2(-0.1, 0.1) = 0$ ✓

$n_3(-0.1, -0.1) = 0$ ✓ $\quad n_3(0.1, -0.1) = 0$ ✓

$n_3(0.1, 0.1) = 1$ ✓ $\quad n_3(-0.1, 0.1) = 0$ ✓

$$\frac{\partial n_2}{\partial X} = \frac{1}{4}(10 - 100Y) \qquad \frac{\partial n_2}{\partial Y} = \frac{1}{4}(-10 - 100X)$$

$$\frac{\partial n_3}{\partial X} = \frac{1}{4}(10 + 100Y) \qquad \frac{\partial n_3}{\partial Y} = \frac{1}{4}(10 + 100X)$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\,\underset{\sim}{d}$$

$$\begin{bmatrix} \varepsilon_{XX} \\ \varepsilon_{YY} \\ \gamma_{XY} \end{bmatrix} = \begin{bmatrix} \frac{\partial n_2}{\partial X} & 0 & \frac{\partial n_3}{\partial X} & 0 \\ 0 & \frac{\partial n_2}{\partial Y} & 0 & \frac{\partial n_3}{\partial Y} \\ \frac{\partial n_2}{\partial Y} & \frac{\partial n_2}{\partial X} & \frac{\partial n_3}{\partial Y} & \frac{\partial n_3}{\partial X} \end{bmatrix} \begin{bmatrix} d_{2X} \\ d_{2Y} \\ d_{3X} \\ d_{3Y} \end{bmatrix}$$

4) (a) (cont.) However $d_{2Y} = 0$, hence:

$$\underset{\sim}{\varepsilon} = \frac{10}{4} \underbrace{\begin{bmatrix} 1-10Y & 1+10Y & 0 \\ 0 & 0 & 1+10X \\ -1-10X & 1+10X & 1+10Y \end{bmatrix}}_{\underset{\sim}{B}} \begin{bmatrix} d_{2X} \\ d_{3X} \\ d_{3Y} \end{bmatrix}$$

(b)

$$\underset{\sim}{K} = \int_V \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, dV \quad \text{where} \quad \underset{\sim}{D} = E \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\nu = 0 \quad \therefore \quad \underset{\sim}{D} = E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$d_{3X} = d_{3Y} = 0$ hence

$$\underset{\sim}{B} = \frac{5}{2} \begin{bmatrix} 1-10Y \\ 0 \\ -1-10X \end{bmatrix}$$

$$\underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} = \frac{5}{2} \begin{bmatrix} 1-10Y & 0 & -1-10X \end{bmatrix} E \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \frac{5}{2} \begin{bmatrix} 1-10Y \\ 0 \\ -1-10X \end{bmatrix}$$

$$= \frac{25E}{4} \begin{bmatrix} 1-10Y & 0 & \frac{-1-10X}{2} \end{bmatrix} \begin{bmatrix} 1-10Y \\ 0 \\ -1-10X \end{bmatrix}$$

$$= \frac{25E}{4} \left\{ (1-10Y)^2 + \frac{1}{2}(-1-10X)^2 \right\}$$

4) (b) (cont.)

$$\underset{\sim}{K} = t\int_A \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, dA$$

$$= \frac{25}{4}Et \int_{-0.1}^{0.1}\int_{-0.1}^{0.1} 1 - 20Y + 100Y^2 + \tfrac{1}{2}\left(1 + 20X + 100X^2\right) dX\, dY$$

$$= \frac{25}{4}Et \int_{-0.1}^{0.1} \left[X - 20XY + 100XY^2 + \tfrac{1}{2}X + \frac{10X^2}{2} + \frac{50X^3}{3}\right]_{X=-0.1}^{X=0.1} dY$$

$$= \frac{25}{4}Et \int_{-0.1}^{0.1} 0.2 - 4Y + 20Y^2 + 0.1 + \frac{0.1}{3} \quad dY$$

$$= \frac{25}{4}Et \int_{-0.1}^{0.1} \tfrac{1}{3} - 4Y + 20Y^2 \; dY$$

$$= \frac{25}{4}Et \left[\tfrac{1}{3}Y - 2Y^2 + \frac{20Y^3}{3}\right]_{-0.1}^{0.1}$$

$$= \frac{25}{4} \times 100 \times 10^9 \times 10^{-3} \left\{\tfrac{1}{3} \times 0.2 + 2 \times \frac{20}{3} \times 0.1^3\right\}$$

$$= 50 \times 10^6 \text{ N/m}$$

$$= 50 \text{ kN/mm}$$