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SECTION ANSWERS

0.1 Answer to question 1

(a) In the least squares criterium we consider the sum of squared errors (SSE):

$$S = \sum_{i=1}^{n} r_i^2 = \sum_{i=1}^{n} (y_i - (a + bx_i))^2$$

In order to find the parameters a and b that minimize the sum of squared errors we need to differentiate the SSE with respect to the parameters a and b. Setting $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$ gives us the minimum of the SSE.

We start by differentiating the SSE with respect to a

$$\frac{\partial S}{\partial a} = \sum_i 2(a + bx_i - y_i)$$

Setting this to zero we find

$$0 = an + bn\bar{x} - n\bar{y}$$

or

$$\bar{y} = a + b\bar{x}$$

Thus the line of best fit passes through the average point $(\bar{x}, \bar{y})$. This makes sense intuitively and also hints towards the fact that there might be no bias in the parameters. We will shortly see that this is indeed the case.

Next differentiate the SSE with respect to b

$$\frac{\partial S}{\partial b} = \sum_i 2(a + bx_i - y_i)x_i$$

Again, setting this to zero and using equation (a) we get

$$0 = \sum_i (\bar{y} - b\bar{x} + bx_i - y_i)x_i$$

$$\sum_i (y_i - \bar{y})x_i = b\sum_i (x_i - \bar{x})x_i$$

Thus we find

$$b = \frac{\sum_i x_i(y_i - \bar{y})}{\sum_i x_i(x_i - \bar{x})} = \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sum_i x_i x_i - n\bar{x}^2}$$

i.e. (using the hint)

$$b = \frac{14983.07 - 2206.52 * 161.5/24}{203568.55 - 2206.52^2/24} = 0.19159$$

Once we have b we can find a

$$a = \bar{y} - b\bar{x}$$

i.e.

$$a = 161.5/24 - 0.19159 * 2206.52/24 = -10.885$$

(b) The correlation coefficient R is a measure of the strength of the relationship between the independent and the dependent variables, i.e. how well does the right hand side of the equation explain the left hand side.

It is defined by

$$R = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 \sum_i (y_i - \bar{y})^2}} = \frac{\sum_i x_i y_i - n\bar{x}\bar{y}}{\sqrt{(\sum_i x_i^2 - n(\bar{x})^2)(\sum_i y_i^2 - n(\bar{y})^2)}}$$

i.e.

$$R = \frac{14983.07 - 2206.52 * 161.50/24}{\sqrt{(203568.55 - 2206.52^2/14)(1120.75 - 161.5^2/24)}} = 0.8724$$

The bigger $|R|$ is, the stronger the linear relationship between the dependent and the independent variables is. Our value for R is rather large so there is a strong correlation between the two data.

(c) A measure of the accuracy of the prediction obtained from a regression model is given by the standard deviation of the estimation errors. The standard error measures the scatter in the actual data around the estimated regression line.

$$S_e = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-2}}$$

It is an unbiased estimator for the parameter σ.

(d) The simple linear regression model is

$$y = \alpha + \beta x + \varepsilon$$

α and β are parameters of the model, ε is a random variable called the error term. Our aim was to find parameters a and b such that the diviations between model predictions

$$\hat{y}_i = a + bx_i$$

and observations y_i are small at all data points. Thus a and b are not necessarily equal to α and β respectively but rather they are estimates.

$$a = \hat{\alpha}, \qquad b = \hat{\beta}$$

. The estimates are unbiased so that the mean of a equals α and the mean of b equals β.

According to the central limit theorem, when n is large enough, b has a normal distribution with mean β and variance σ^2/S_{XX} where σ^2 is the unknown variance of the ε.

$$b \sim N\left(\beta, \frac{\sigma^2}{S_{XX}}\right)$$

Further a can be written as $\bar{y} - b\bar{x}$ and thus is a sum of two independent normal variables. Therefore it is itself normal with mean being the sum of the means and the variance being the sum of the variances of the two variables.

$$Var(a) = \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}}\right)\sigma^2$$

and thus

$$a \sim N\left(\alpha, \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{XX}}\right)\sigma^2\right)$$

Since we don't know σ we use the standard error S_e to estimate it.

0.2 Answer to question 2

(a) Weights for the the MVP are given by

$$\omega_k = \frac{\sum_i \sigma_{ki}^{-1}}{\sum_k \sum_i \sigma_{ki}^{-1}}$$

$$\sum_i (\sigma^{-1})_{ij} = (3.24, 1.62, 1.08, 0.81)$$

$$\sum_{ij} (\sigma^{-1})_{ij} = 6.7568$$

Hence

$$\omega_k = (0.48, 0.24, 0.16, 0.12)$$

The expected return of the MVP portfolio is

$$\omega_k r_k = 0.0692$$

and the variance is

$$\sigma^2_{MVP} = \frac{1}{\sum_k \sum_i \sigma^{-1}_{ki}} = 1/6.7568 = 0.148$$

(b) To find the tangent point that represents the efficient fund we again need to solve an optimization problem. The slope of the line passing through r_f and P is given by

$$\theta = \frac{r_P - r_f}{\sigma_P}$$

and clearly the tangent portfolio is the one that maximizes this slope. In terms of the weights of the portfolio P this equals

$$\theta = \frac{\sum_i \omega_i (r_i - r_f)}{\sqrt{\sum_{i,j} \sigma_{ij} \omega_i \omega_j}}$$

Differentiating with respect to the weights yields the equations

$$\lambda \sum_i \sigma_{ji} \omega_i = r_j - r_f$$

Solving this yields

$$\omega_i \propto \sum_j \sigma^{-1}_{ij} (r_j - r_f)$$

In our case

$$\omega_i \propto (-0.0297, 0, 0351, 0.568, 0, 0676)$$

All that remains is to normalize the weights such to $\sum_i \omega_i = 1$. Normalizing, we find

$$\omega_i = (-0.23, 0.27, 0.44, 0.52)$$

(c) If all investors are mean-variance portfolio optimizers and they share the same expectation as to expected returns, variances and covariances. By the one fund theorem every investor will hold some amounts of the riskless asset and the same portfolio of risky assets. The mix of these two assets will be likely to vary across investors depending on their risk preference. However, every investor will hold some proportion of the one-fund portfolio.

Since all assets must be held by somebody, this implies that every investor holds the market portfolio, which is the summation of all assets available. If everyone buys only one fund and their purchases add up to the market, then that fund must be the the market as well. In the world of shares the market portfolio contains shares of every stock in proportion to that stock's representation in the entire market. The weight ω_i of an asset is equal to the proportion of that asset's total capital value to the total market capital value.

Our portfolio P cannot be a market portfolio since one of the assets in P is shorted.

(d) Since we calculated the weights for both P and the MVP-portfolio we can construct any portfolio on the efficient frontier by taking a linear combination of their weights (two-fund theorem).

$$\omega_i = \alpha(0.48, 0.24, 0.16, 0.12) + (1-\alpha)(-0.23, 0.27, 0.44, 0.52)$$

Only asset 1 can be shorted in this linear combination (assuming $\alpha \geq 0$ of course), so we find the value for α for which the weight $\omega_1 = 0$.

$$(0.48 + 0.23)\alpha = 0.23 \implies \alpha = 0.32$$

The weights of the portfolio with largest expected return, that lies on the efficient frontier and has no asset shorted are

$$\omega_i = (0, 0.26, 0.35, 0.39)$$

0.3 Answer to question 3

(a) Patients will arrive independently and their arrival is a Poisson process.

(b) We again have a random variable $N(t)$ that represents the number of patients in the system, but we now have births (arrivals) and deaths (being seen by the nurse) that can both increase and decrease this number.

Given time t and state $N(t) = n$, the probability distribution of the remaining time until the next birth is exponential with parameter λ.

Given time t and state $N(t) = n > 0$, the probability distribution of the remaining time until the next death is exponential with parameter μ_N.

We will be interested in the steady state behaviour of the process. The process is in steady state if $p_n = P(N(t) = n)$ is independent of the time t.

At any time the next state transition is either from n to $n+1$ or from n to $n-1$, depending on whether a birth or a death occurs first.

Define E_n and L_n to be the rate (average number of events per unit time) at which the system enters and leaves state n respectively.

Then for steady state we have the balance equation

$$E_n = L_n$$

The process can only enter state 0 from state 1. p_1 is the proportion of time the process is in state 1 and μ_N is the rate at which the process enters state 0 from state 1.

Hence

$$E_0 = \mu_N p_1, \quad L_0 = \lambda p_0$$

Similar arguments show that

$$\begin{aligned} E_n &= p_{n-1}\lambda + p_{n+1}\mu_N \\ L_n &= p_n(\lambda + \mu_N) \end{aligned}$$

Using the balance equation $E_0 = L_0$ we obtain

$$p_1 = \frac{\lambda}{\mu_N} p_0$$

Using the balance equations $E_n = L_n$ we further obtain

$$p_{n+1} = \frac{\lambda + \mu_N}{\mu_N} p_n - \frac{\lambda}{\mu_N} p_{n-1}$$

The solution of this recursion relation is

$$p_n = c_n p_0$$

where

$$c_n = \frac{\lambda^n}{\mu_N^n}$$

Since

$$\begin{aligned} 1 &= \sum_{i=0}^{\infty} p_i \\ &= \sum_{i=0}^{\infty} c_i p_0 \\ &= p_0 \sum_{i=0}^{\infty} c_i \end{aligned}$$

Hence

$$p_0 = 1 - \frac{\lambda}{\mu_N}$$

and

$$p_n = \left(\frac{\lambda}{\mu_N}\right)^n \left(1 - \frac{\lambda}{\mu_N}\right)$$

(c) The average number of people in the queue is given by

$$L_q = \sum_i i p_{i+1} = \frac{\lambda}{\mu_N} \frac{\lambda}{\mu_N - \lambda}$$

(d) By Little's formula

$$W_q = L_q/\lambda = \frac{1}{\mu_N} \frac{\lambda}{\mu_N - \lambda}$$

(e) The average time until patients are seen by the nurse is

$$W_{(N)} = W_q + \frac{1}{\mu_N} = \frac{1}{\mu_N - \lambda}$$

After been seen by the nurse, patients are put into another queue to see the doctor. The output from the nurse is also a Poisson process of rate λ. By repeating the arguments above, the average time until the patient is then seen by the doctor is

$$W_{(D)} = \frac{1}{\mu_D - \lambda}$$

Thus the average total time taken for a patient to see both the nurse and the doctor is

$$W = \frac{1}{\mu_N - \lambda} + \frac{1}{\mu_D - \lambda}$$

(f) The first term is smaller than the second one (since $\mu_D < \mu_N$), thus in order to speed up the time taken, it would be better to increase the number of doctors.

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0.4 Answer to question 4

(a) There are 3 classes:

•A:{1,3,6} closed

•B:{2,4} not closed

•C:{5} closed

(b) Since A is closed we can look at the submatrix of transitions amongst states in A.

$$P' = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Then solve for $uP' = u$. This gives $u \propto (8,15,12)$. Normalizing we find $u = (8/35, 15/35, 12/35)$.

$$P(state\ 6, n \to \infty) = \frac{12}{35}$$

(c) Let h_{ij} be the probability that we ever hit state j, starting from state i. We can use the relation

$$h_{ij} = \sum_k p_{ik} h_{kj}$$

Clearly $h_{46} = h_{4A}$ since once we hit any state in class A we will eventually hit state 6 as A is closed. $h_{AA} = 1$ and $h_{5A} = 0$.

$$h_{4A} = p_{4A}h_{AA} + p_{42}h_{2A} + p_{44}h_{4A} + p_{45}h_{5A}$$

$$h_{4A} = 2/5 + 1/5h_{2A} + 1/5h_{4A} + 0$$

and

$$h_{2A} = p_{2A}h_{AA} + p_{22}h_{2A} + p_{24}h_{4A} + p_{25}h_{5A}$$

$$h_{2A} = 3/6 + 1/6h_{2A} + 1/6h_{4A} + 0$$

Combining these two:

$$
\begin{aligned}
4h_{4A} - h_{2A} &= 2 \\
h_{4A} - 5h_{2A} &= -3
\end{aligned}
$$

we find

$$h_{4A} = \frac{13}{19}$$