Prof M C Smith

3F1 2008 Solution

1 (a) Using the time advance property of the z-transform we get

$$zY(z) - zy_0 + aY(z) = bU(z)$$

which can be rearranged to

$$Y(z)(z+a) = bU(z) + \beta z$$

Dividing both sides by $z+a$ gives

$$Y(z) = \frac{b}{z+a}U(z) + \frac{\beta z}{z+a}$$

which is the desired result. [15%]

(b) We can solve recursively the difference equation

$$\begin{aligned} y_0 &= \beta \\ y_1 &= -a\beta \\ y_2 &= a^2\beta \\ y_3 &= -a^3\beta \\ &\vdots \end{aligned}$$

giving a general solution $y_k = (-a)^k \beta$. From left to right are the graphs of y_k versus k for $a < -1$, $-1 < a < 0$, $a = 0$, $0 < a < 1$ and $a > 1$, respectively. [30%]

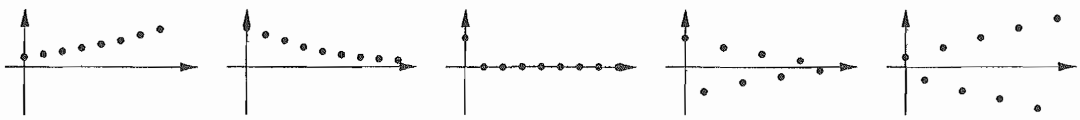


Fig. 1

(c) Substituting the feedback law into the difference equation gives $y_{k+1} + ay_k = b(-cy_k + r_k)$, or $y_{k+1} + (a+bc)y_k = br_k$. Taking the z-transform on both sides, with zero initial condition, gives

$$T(z) = \frac{Y(z)}{R(z)} = \frac{b}{z+a+bc}$$

The closed-loop system is stable if and only if the pole at $-(a+bc)$ is inside the open unit disk, i.e., if $|a+bc| < 1$. [20%]

(d) We have that

$$\left|T(e^{j\theta})\right| = \frac{|b|}{\left|e^{j\theta}+a+bc\right|}.$$

Using the triangle inequality

$$\left|e^{j\theta}+a+bc\right| \le \left|e^{j\theta}\right| + |a+bc| = 1+|a+bc| \le 2.$$

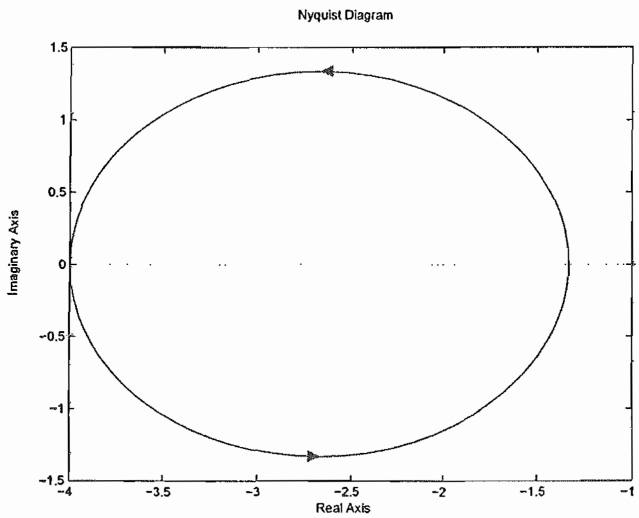
and

$$\left|e^{j\theta}+a+bc\right| \ge 1-|a+bc|.$$

from which the result follows. [35%]

2 (a) (i) The closed-loop system will be stable if and only if the Nyquist diagram of $G(z)$ encircles the $-1/k$ point P times counterclockwise where P is the number of poles of $G(z)$ outside the unit circle. [10%]

(ii) The Nyquist diagram of $G(z)$ is shown below. [20%]



(iii) For $k = 0.5$, $-1/k = -2$ and there is one counterclockwise encirclement, so closed-loop is stable. For $k = 2$, $-1/k = -0.5$ and there is no encirclement, so closed-loop is unstable. [20%]

2 (b) (i) $\Phi_X(u) = \int_{-\infty}^{\infty} e^{jux} f_X(x)\, dx$

$$= \int_0^{\infty} e^{jux} . a e^{-ax}\, dx = \int_0^{\infty} a\, e^{(ju-a)x}\, dx$$

$$= \left[\frac{a}{ju-a} e^{(ju-a)x} \right]_0^{\infty} = 0 - \frac{a}{ju-a} = \underline{\underline{\frac{a}{a-ju}}}$$

[20%]

(ii) $\frac{d\Phi_X(u)}{du} = \int_{-\infty}^{\infty} (j \cdot x) . e^{jux} f_X(x)\, dx$

$$\therefore \frac{d^n\Phi_X(u)}{du^n} = \int_{-\infty}^{\infty} (jx)^n . e^{jux} f_X(x)\, dx$$

$$= j^n \int_{-\infty}^{\infty} x^n . e^{jux} f_X(x)\, dx$$

(so that $e^{jux} = 1$)

Hence evaluating this at $u=0$ / & dividing by j^n gives:

$$\frac{1}{j^n} . \left. \frac{d^n\Phi_X(u)}{du^n} \right|_{u=0} = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

$= n^{th}$ moment of $X = E\{X^n\}$

For the given pdf $f_X(u)$:

1^{st} moment: $\frac{d\Phi}{du} = \frac{d}{du}\left(\frac{a}{a-ju}\right) = \frac{+ja}{(a-ju)^2}$

Hence $E\{X\} = \frac{1}{j} . \left. \frac{ja}{(a-ju)^2} \right|_{u=0} = \frac{ja}{ja^2} = \underline{\underline{\frac{1}{a}}}$

2^{nd} moment: $\frac{d^2\Phi}{du^2} = \frac{ja}{(a-ju)^3} . 2j = \frac{-2a}{(a-ju)^3}$ [30%]

$\therefore E\{X^2\} = \frac{1}{j^2} . \left. \frac{-2a}{(a-ju)^3} \right|_{u=0} = \frac{-2a}{-a^3} = \underline{\underline{\frac{2}{a^2}}}$

3 (a) When independent random variables X and Y are added together their pdfs are convolved together so that, if $Z = X + Y$

$$f_Z(z) = f_X(x) * f_Y(y)$$

If X & Y are both Gaussian then Z will also be a Gaussian process whose mean is $\bar{X} + \bar{Y}$ and whose variance $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$. [20%]

(b)(i) $r_{xx}(\tau) = E\{X(t)\,X(t+\tau)\}$

The mean power of X into a unit impedance load is $E\{x^2(t)\}$.

Hence, mean power of $X = r_{xx}(0) = A$ [10%]

(ii) $$Y(t) = h(t) * X(t)$$
$$= \delta(t) * X(t) + \delta(t-T) * X(t)$$
$$= X(t) + X(t-T)$$

$$\therefore r_{YY}(\tau) = E\{Y(t)\,Y(t+\tau)\}$$

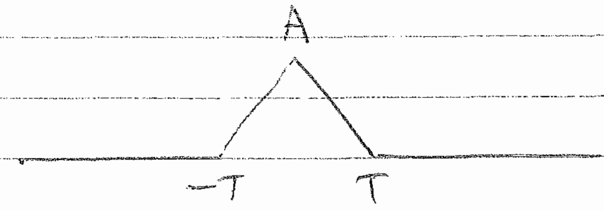
$$= E\{[X(t) + X(t-T)][X(t+\tau) + X(t-T+\tau)]\}$$

$$= E\{X(t)X(t+\tau)\} + E\{X(t)X(t-T+\tau)\}$$
$$+ E\{X(t-T)X(t+\tau)\} + E\{x(t-T)X(t-T+\tau)\}$$

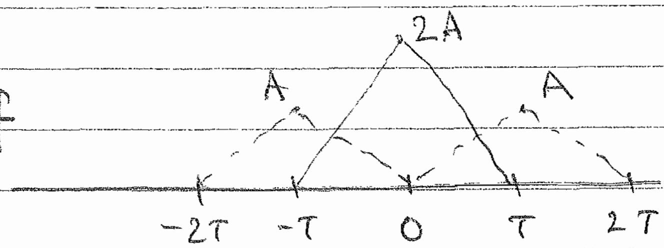
$$= r_{xx}(\tau) + r_{xx}(\tau - T) + r_{xx}(\tau + T)$$
$$+ r_{xx}(\tau)$$
$$= r_{xx}(\tau - T) + 2r_{xx}(\tau) + r_{xx}(\tau + T)$$

3(b)(ii) (cont.)

$r_{xx}(\tau)$



$r_{yy}(\tau)$ = sum of



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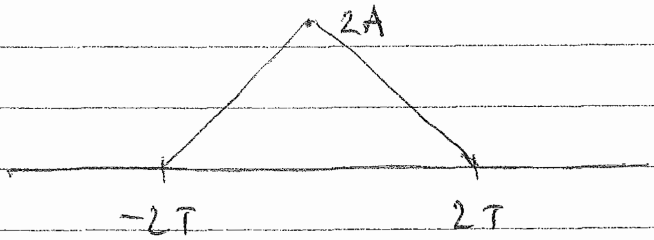

Hence $r_{yy}(\tau) = \begin{cases} 2A\left(1 - \frac{|\tau|}{2T}\right) & \text{for } |\tau| \le 2T \\ 0 & \text{for } |\tau| > 2T \end{cases}$

Mean power of $Y(t) = r_{yy}(0) = 2A$ [50%]

(iii) $Y(t)$ is the sum of $X(t)$ and $X(t-T)$ so its pdf will also be Gaussian, since they are assumed to be independent.

$X(t)$ is zero mean so its power will equal its variance: $\sigma_x^2 = A$.

Hence $Y(t)$ has zero mean and variance $2A$.

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2/(2\sigma_y^2)}$$

$$= \frac{1}{\sqrt{4\pi A}} e^{-y^2/(4A)}$$ [20%]

4(a) Multiplying each inequality by p_i and summing over i gives:

$$\sum_i p_i \log_2\left(\frac{1}{p_i}\right) \leq \sum_i p_i l_i < \sum_i p_i \log_2\left(\frac{1}{p_i}\right) + \sum_i p_i$$

Since $H(S) = \sum_i p_i \log_2\left(\frac{1}{p_i}\right)$ and $L = \sum_i p_i l_i$ this is the required result.

[25%]

(b)(i) $P(A) = P(A,A) + P(A,B)$
$= 0.84 + 0.06 = 0.9$

$P(B) = P(B,A) + P(B,B)$
$= 0.06 + 0.04 = 0.1$

$$P(X_n | X_{n-1}) = P(X_n, X_{n-1}) / P(X_{n-1})$$

Conditional probability table:

$P(X_n | X_{n-1})$:

X_n \ X_{n-1}	A	B
A	0.9333	0.6
B	0.0667	0.4

[20%]

(ii) Conditional entropy: $H(X_n | X_{n-1})$

$$= H(X_n | X_{n-1} = A) P(A) + H(X_n | X_{n-1} = B) P(B)$$

4(b)(ii) (cont.)

$$= \left[0.9333 \log_2 \frac{1}{0.9333} + 0.0667 \log_2 \frac{1}{0.0667}\right] 0.9$$

$$+ \left[0.6 \log_2 \frac{1}{0.6} + 0.4 \log_2 \frac{1}{0.4}\right] 0.1$$

$$= [.0929 + 0.2605]\,0.9 + [0.4422 + 0.5288]\,0.1$$

$$= 0.4151$$

Mutual information

$$I(X_n ; X_{n-1}) = H(X_n) - H(X_n | X_{n-1})$$

$$= \left[0.9 \log_2 \frac{1}{0.9} + 0.1 \log_2 \frac{1}{0.1}\right] - 0.4151$$

$$= [0.1368 + 0.3322] - 0.4151$$

$$= 0.0539$$

Total entropy for N consecutive symbols: $H(X_1, \ldots, X_n)$

$$= H(X_1) + H(X_2 | X_1) + \ldots + H(X_N | X_{N-1})$$

$$= H(X_n) + (N-1) H(X_N | X_{N-1})$$

$$= I(X_N ; X_{N-1}) + N\, H(X_N | X_{N-1})$$

$$= 0.0539 + N\, 0.4151$$

[40%]

(iii) From part (ii), with $H(S) = H(X_1, \ldots, X_n)$,

$$H(S) \leq L < H(S) + 1$$

$$\Rightarrow \frac{H(S)}{N} \leq \frac{L}{N} < \frac{H(S)+1}{N}$$

Hence $0.4151 + \dfrac{0.0539}{N} \leq \dfrac{L}{N} < 0.4151 + \dfrac{1.0539}{N}$

[15%]