3F2 Systems and Control: 2008 Solutions

Prof. J.M. Maciejowski

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1. (a) *Construction of root-locus diagram:* The return-ratio is $L(s) = kF/s(Js + B)$. So there are 2 poles, at 0 and at $-B/J$, and no zeros. Thus there are 2 branches of the root-locus, which start at the poles, and both branches become infinite as $k \to \infty$. Every point on the real axis between the two poles is on the real axis (as a consequence of the *angle criterion*). Since the excess of poles over zeros is 2, the 2 asymptotes are parallel to the imaginary axis. They are located at the 'centre of gravity' of the poles and zeros: $\frac{0-B/J}{2-0} = -\frac{B}{2J}$. Simple geometry, using the angle criterion

$$\arg(s_0 - 0) + \arg(s_0 + B/J) = \pi \tag{1}$$

shows that the root-loci in fact coincide with the asymptotes. Thus the root-locus diagram is as shown in Fig.1.

Alternative method: The root-locus is the locus of solutions of the characteristic equation

$$1 + L(s) = 0 \tag{2}$$

$$\text{namely} \quad 1 + \frac{kF}{s(Js+B)} = 0 \tag{3}$$

$$\text{or} \quad s^2 + \frac{B}{J}s + \frac{kF}{J} = 0. \tag{4}$$

This has solutions

$$s = -\frac{B}{2J} \pm \sqrt{\left(\frac{B}{J}\right)^2 - 4\frac{kF}{J}} \tag{5}$$

which shows that for small k the roots are real and symmetrically located about $-B/2J$, and for large k the real part is $-B/2J$ and the imaginary part increases monotonically with k (and eventually proportionally to $\sqrt{k}$).

Interpretation of root-locus diagram: The diagram shows that the closed loop is stable for all $k > 0$. For small values of k both closed-loop poles are real; as k is increased in this range the closed-loop transient response becomes faster, because the 'slower' pole becomes 'faster' (moves to the left). Beyond some value of k (which can easily be calculated to be $B^2/4JF$, but this is not expected) the closed-loop poles become complex. The real part thereafter stays the same, meaning that the transient response envelope time constant does not change, but the imaginary part becomes larger as k increases, meaning that the response becomes increasingly underdamped, with oscillations of increasing frequency.

(b) The quick way to do this is to consider steady-state conditions only, when all signals have settled down to constant values, assuming that the closed loop is asymptotically stable. Define the position error to be $e = \theta_d - \theta$. If the demanded angle is to be achieved exactly then in the steady-state we must have $e = 0$, and θ must be constant. But the motor includes an integration, so its output can only be constant if its input is 0. Hence we must have

$$i = -\frac{\tau_L}{F} \neq 0 \tag{6}$$

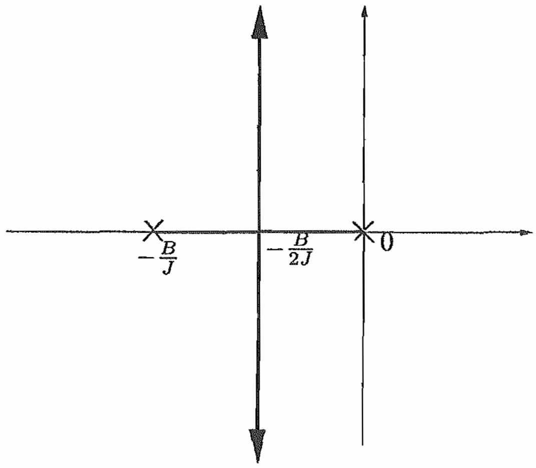

Figure 1: Root-locus diagram for Q.1(a).

So the controller must have input $e = 0$ and output $i \neq 0$ in the steady state. But its steady-state gain is $K(0)$, which must therefore be infinite. The only way of achieving that is to include integral action in the controller.

More formal approach using Final Value Theorem: From the block diagram

$$E(s) = \Theta_d(s) - \frac{F}{s(Js+B)}\left[\frac{\tau_L(s)}{F} + K(s)E(s)\right]. \tag{7}$$

Hence

$$\left[1 + \frac{K(s)F}{s(Js+B)}\right]E(s) = \Theta_d(s) - \frac{\tau_L(s)}{s(Js+B)} \tag{8}$$

or

$$E(s) = \frac{s(Js+B)\Theta_d(s) - \tau_L(s)}{s(Js+B) + K(s)F}. \tag{9}$$

Now by the Final Value Theorem, assuming closed-loop asymptotic stability, $\Theta_d(s) = \theta_d/s$ and $\tau_L(s) = \tau_L/s$:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = -\frac{\tau_L}{K(0)F}. \tag{10}$$

Hence $\lim_{t\to\infty} e(t) = 0$ only if $K(0)$ is infinite, which requires integral action.

Note: The manipulations above become simpler if a generic transfer function $G(s)$ is used in place of the motor transfer function. Quoting (or deriving) the result $E(s) = G(s)\tau(s)/F[1 + G(s)K(s)]$ (assuming $\theta_d = 0$ for simplicity) would then be acceptable.

(c) The return-ratio is now

$$L(s) = \frac{F}{s(Js+B)} \times \frac{k(s+a)}{s} \tag{11}$$

There are now 3 poles and 1 zero. One branch of the root-locus diagram is real and lies between the pole at $-B/J$ and the zero at $-a$; hence it remains negative since $a > 0$. Two branches of the root-locus diagram become infinite as $k \to \infty$; the asymptotes are similar to those in part (a), namely they emanate from the real axis at angles $\pm\pi/2$. The location of the asymptotes depends on the value of a.

The complete root-locus diagram looks something like Fig.2 when $a < B/J$. This is the only possibility since the assumption is that there are no breakaway points other than the origin. Otherwise a diagram like Fig.3 would be possible. If $a > B/J$ then the diagram is similar to Fig.2 except that the asymptotes, and the complex branches of the locus, are in the right half-plane.

(d) Just reading off the root-locus diagram (Fig. 2) is almost sufficient, except that we have not proved that the loci never go into the right half-plane before approaching the asymptotes.

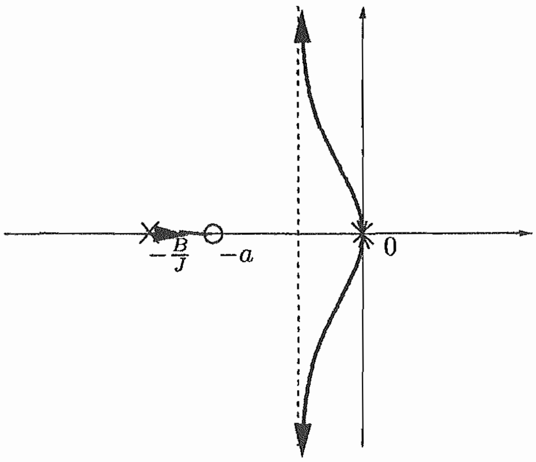

Figure 2: Root-locus diagram for Q.1(c).

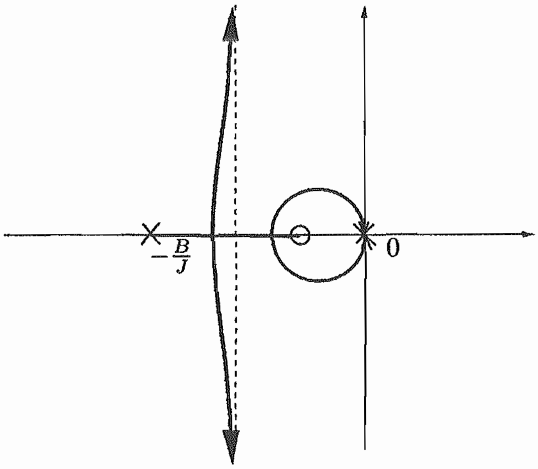

Figure 3: Possible root-locus diagram for Q.1(c) if breakaway points other than the origin existed.

Method 1, using root-locus arguments: Consider a point s_0 in the complex plane, as shown in Fig.4. Let $\arg(s_0) = \theta$, $\arg(s_0 + a) = \alpha$, and $\arg(s_0 + B/J) = \beta$, as shown in Fig.4. By the angle criterion, s_0 is on the root-locus only if

$$2\theta - \alpha + \beta = \pi \tag{12}$$

But if s_0 is in the right half-plane then $\theta < \pi/2$, and if $a < B/J$ then $\alpha > \beta$, and hence

$$2\theta - \alpha + \beta < \pi \tag{13}$$

so s_0 cannot be on the root-locus. Hence the root-locus must lie entirely in the left half-plane (for $k > 0$), and hence the closed loop must be stable for all $k > 0$.

Method 2, using the Routh-Hurwitz criterion: Stability is determined by the closed-loop characteristic equation:

$$1 + L(s) = 1 + \frac{F}{s(Js + B)} \times \frac{k(s + a)}{s} = 0 \tag{14}$$

Its roots are the same as the roots of

$$s^2(Js + B) + kF(s + a) = 0 \tag{15}$$

or

$$Js^3 + Bs^2 + kFs + kFa = 0 \tag{16}$$

By the Routh-Hurwitz criterion, all the roots of this have negative real parts if and only if $BkF > JkFa$, namely if $a < B/J$.

Note: Part (d) could also be tackled using Bode plot methods from the second-year course. Such solutions are entirely legitimate and would be given appropriate credit.

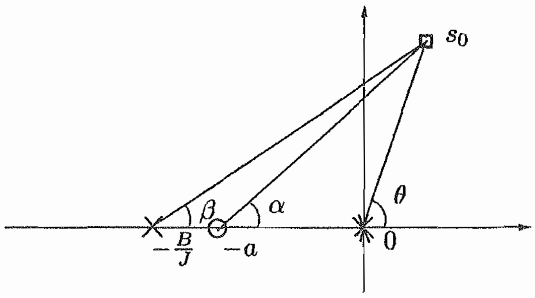

Figure 4: Checking the angle criterion for Q.1(d).

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2. (a) Let n be the dimension of the state vector x. Construct the controllability matrix:

$$P = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \tag{17}$$

Then the system is controllable if and only if

$$\text{rank}(P) = n \tag{18}$$

(b) Take Laplace transforms of the defining equations in part (a):

$$sX(s) - x(0) = AX(s) + BU(s) \tag{19}$$
$$Y(s) = CX(s). \tag{20}$$

Write $sX(s)$ as $sIX(s)$ and manipulate (19) into the form

$$(sI - A)X(s) = BU(s) + x(0). \tag{21}$$

The transfer function is defined with the assumption that $x(0) = 0$, hence (21) gives

$$X(s) = (sI - A)^{-1}BU(s) \tag{22}$$

Substituting this into (20) gives

$$Y(s) = C(sI - A)^{-1}BU(s) \tag{23}$$

and hence the transfer function (matrix) is

$$G(s) = C(sI - A)^{-1}B \tag{24}$$

(c) In this case $n = 2$ so the controllability matrix is

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -3+b \\ b & -2b \end{bmatrix} \tag{25}$$

Hence its determinant is

$$\det(P) = -2b - b(-3+b) = b - b^2 = b(1-b). \tag{26}$$

The system is not controllable when P loses rank, namely when $\det(P) = 0$. By inspection of (26) it can be seen that this occurs for:

$$b = 0 \quad \text{and} \quad b = 1.$$

(d) Applying (24), the transfer function is given by

$$G(s) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} 1 \\ b \end{bmatrix}.$$

But

$$(sI - A)^{-1} = \begin{bmatrix} s+3 & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s+3 \end{bmatrix}}{(s+3)(s+2)}$$

hence

$$\begin{aligned} G(s) &= \frac{\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 0 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix}}{(s+3)(s+2)} \\ &= \frac{\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} s+2+b \\ b(s+3) \end{bmatrix}}{(s+3)(s+2)} \\ &= \frac{c_1(s+2+b) + c_2 b(s+3)}{(s+3)(s+2)} \end{aligned}$$

As expected with a 2-state model, this is a 2nd-order transfer function. However at the two values of b for which controllability is lost, pole-zero cancellations occur:

$$\begin{aligned} b = 0: \quad G(s) &= \frac{c_1(s+2)+0}{(s+3)(s+2)} = \frac{c_1}{s+3}, \\ b = 1: \quad G(s) &= \frac{c_1(s+3)+c_2(s+3)}{(s+3)(s+2)} = \frac{c_1+c_2}{s+2}. \end{aligned}$$

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3. (a) The ODE is second-order, so expect to need 2 state variables. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Then

$$\dot{x}_1 = x_2 \tag{27}$$

$$\dot{x}_2 = -\frac{F}{J}x_1 + \frac{1}{J}u + \frac{1}{J}d \tag{28}$$

which can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{F}{J} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} d \tag{29}$$

which is in the required form, with the state vector x defined as $x = [x_1, x_2]^T$.
The matrix A is triangular; hence its eigenvalues are just its diagonal elements, namely 0 and $-F/J$. The open-loop system is thus marginally stable, with one eigenvalue on the imaginary axis and the other negative (since $F > 0$ and $J > 0$).

The phrase '...using as few state variables as possible' was intended to be helpful rather than sneaky, but it caused some candidates to use only one state variable ($x = \dot{\theta}$), which was ok for this part, but they then had to use a 2-state model in the next part.

(b) Since d is not measured we cannot use it when determining the control signal u. But both state variables are measured, so we can use a state-feedback control of the form

$$u = -k^T x = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x \tag{30}$$

where $k = [k_1, k_2]^T$. Substituting this into the state-space equation we get

$$\dot{x} = Ax + B_1 u + B_2 d \tag{31}$$

$$= Ax - B_1 k^T x + B_2 d \tag{32}$$

$$= (A - B_1 k^T)x + B_2 d \tag{33}$$

The closed-loop dynamics depend on the eigenvalues of the matrix:

$$A - B_1 k^T = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{F}{J} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \quad (34)$$

$$= \begin{bmatrix} 0 & 1 \\ -\frac{k_1}{J} & -\frac{F}{J} - \frac{k_2}{J} \end{bmatrix} \quad (35)$$

The eigenvalues are given by the roots of the characteristic polynomial:

$$\lambda \left(\lambda + \frac{F}{J} + \frac{k_2}{J} \right) + \frac{k_1}{J} \quad (36)$$

To get these roots to both be p, this should be the same as $(\lambda - p)^2$. Comparing coefficients of λ^1 and λ^0 gives

$$\frac{F}{J} + \frac{k_2}{J} = -2p \qquad \text{and} \qquad \frac{k_1}{J} = p^2. \quad (37)$$

Rearranging gives

$$k_1 = p^2 J \qquad \text{and} \qquad k_2 = -2pJ - F. \quad (38)$$

Comment: Large $|p|$ results in large values of both gains, especially k_1. Not surprisingly, large gains are needed to get fast responses. Also for a given p, large gains will be required if the inertia (J) or viscous friction (F) of the robot is large. Note that, since $p < 0$ will be needed for stable operation, $k_2 > 0$ for small values of F but $k_2 < 0$ for large values, and $k_2 = 0$ if $F = 2|p|J$.

(c) Integral action can be introduced into state-feedback schemes by integrating the error in the controlled variable

$$e = \theta_d - \theta \quad (39)$$

which can be done by introducing an additional state variable, whose derivative is the error:

$$\dot{x}_3 = e \quad (40)$$

Then x_3 will be the integral of the error. Now state feedback of the form

$$u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (41)$$

can be used, and the gains chosen to give desired closed-loop pole locations. The state-space model of the system is enhanced by the addition of the new state variable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{F}{J} & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{J} \\ 0 \end{bmatrix} d + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \theta_d \quad (42)$$

(d) Tracking a ramp without error (in the steady state) generally requires a double-integration in the controller. This can be obtained in the state-feedback framework by extending the idea used in part (c). Introduce another new state variable, which will be the integral of x_3, and hence the double-integral of e:

$$\dot{x}_4 = x_3 \quad (43)$$

Now state feedback will take the form

$$u = -\begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (44)$$

Note: This is the obvious follow-on from part (c), but other reasonable suggestions will be given appropriate credit.

4. (a) Structure and purpose of observer: Standard bookwork, standard block diagram, as in lecture notes.

Nearly everyone knew the block-diagram, but surprisingly many candidates either did not say what the purpose *of an observer was, or got it wrong.*

(b) From the block diagram, the state estimate $\hat{x}$ is governed by the ODE

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu - L(y - C\hat{x}) \tag{45}$$

Subtracting this from the given equation $\dot{x} = Ax + Bu$, and using $y = Cx$ and $\dot{e} = \dot{x} - \dot{\hat{x}}$ gives

$$\dot{e} = Ae - LCe = (A - LC)e. \tag{46}$$

Now, either by analogy with $\dot{x} = Ax$ having solution $x(t) = e^{At}x(0)$, or by checking that the solution satisfies the differential equation, or by taking Laplace transforms ($[sI - (A - LC)]\bar{e}(s) = e(0)$), the solution to this ODE is

$$e(t) = \exp\left[(A - LC)t\right]e(0) \tag{47}$$

where $\exp[M]$ denotes the matrix exponential.

(c) Take the pressures $p_1, \ldots, p_N$ and the flows $q_1, \ldots, q_N$ as state variables, assigning them as follows (for convenience): $x_1 = q_1, x_2 = p_1, x_3 = q_2, \ldots, x_{2N} = p_N$. Then the equation for $x_1 = q_1$ is

$$\dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}p_0 \tag{48}$$

which is in state-space form with an external input (p_0). The equation for $x_{2N} = p_N$ is

$$\dot{x}_{2N} = \frac{1}{C}x_{2N-1} - \frac{1}{C}q_{N+1} \tag{49}$$

which is again in state-space form with an external input (q_{N+1}). All the equations for the other states involve states only:

$$\dot{x}_{2k-1} = \dot{q}_k = -\frac{R}{L}x_{2k-1} + \frac{1}{L}(x_{2k-2} - x_{2k}) \qquad (k = 2, \ldots, N) \tag{50}$$

$$\dot{x}_{2k} = \dot{p}_k = \frac{1}{C}(x_{2k-1} - x_{2k+1}) \qquad (k = 1, \ldots, N-1). \tag{51}$$

Note: It is ok to work with p's and q's, without introducing x's. That keeps the indexing simpler. If these equations are assembled into matrices then using the state variable assignment as above gives a 'banded' A matrix. (But it is not necessary to do this.)

$$\dot{x} = \begin{bmatrix} -R/L & -1/L & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1/C & 0 & -1/C & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1/L & -R/L & -1/L & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1/C & 0 & -1/C & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1/L & -R/L & -1/L \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1/C & 0 \end{bmatrix} x + \begin{bmatrix} 1/L & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1/C \end{bmatrix} \begin{bmatrix} p_0 \\ q_{N+1} \end{bmatrix} \tag{52}$$

The number of state variables required is $2N$.

(d) Now $N = 2$, so with $C = L = 1$ and $R = 2$ we have

$$A = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{53}$$

Since $p_1 = x_2$ is measured, the 'C' matrix is $[0, 1, 0, 0]$. Hence the observability matrix is

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & -2 & 2 & 1 \\ 2 & 4 & -1 & -2 \end{bmatrix} \tag{54}$$

The system is observable if $\det Q \neq 0$. Expanding the determinant by the first row gives

$$\det Q = -1 \times \begin{vmatrix} 1 & -1 & 0 \\ -2 & 2 & 1 \\ 2 & -1 & -2 \end{vmatrix} = -1[1(-4+1) + 1(4-2)] = 1 \neq 0 \tag{55}$$

so the system is observable from measurements of p_1 alone (since it is given that p_0 and q_3 are measured, hence available to the observer as inputs). So all the states can be estimated.

(e) If p_0 and q_{N+1} are not known, but are known to be constant, then they can be estimated by including them among the state variables, with governing equations $\dot{p}_0 = 0$ and $\dot{q}_{N+1} = 0$. Then a state observer for the new system (with $2N + 2$ states) can be used to estimate the states, thus including estimates of p_0 and q_{N+1}, providing that the new system is observable from the available measurements.

Several candidates suggested that p_0 and q_{N+1} could be deduced from $\hat{x}$, once it had converged to x, and the whole system had settled to steady-state (which presupposes that the system is stable, which may not be the case). The problem is that if p_0 and q_{N+1} are not known then the observer considered in part (d) will not work any more, because that needs p_0 and q_{N+1} as inputs.