ENGINEERING TRIPOS PART IIA

Thursday 24 April 2008 9.00 to 12.00

Module 3A1

FLUID MECHANICS I

Answer not more than five questions.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 3A1 Data Sheet for Applications to External Flows (2 pages) and Incompressible Flow Data Card (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A pair of vortices with the *same* circulation, Γ, are located initially as shown in Fig. 1 in an otherwise still fluid.

(a) At this instant of time write down the complex potential for this arrangement. [20%]

(b) Find any stagnation points in the flow and sketch the pattern at this instant. [20%]

(i) This flow is unsteady. Explain (briefly) why. [10%]

(ii) The pair rotates about the origin. Find the angular velocity of this rotation, $d\beta/dt$, where β is the angle of a line joining one vortex to the origin with the x-axis. [10%]

(c) A sink of strength m is now added at the origin causing the vortices to spiral inwards. Find the rate at which a, (the distance of either vortex from the origin), changes, i.e, find da/dt. [20%]

(d) The vortices spiral towards the origin. Find the equation of this spiral trajectory, i.e. the trajectory may be written as $a = f(\beta)$ where f is some function. Find the function $f(\beta)$. [20%]

Note: f depends also on m, Γ.

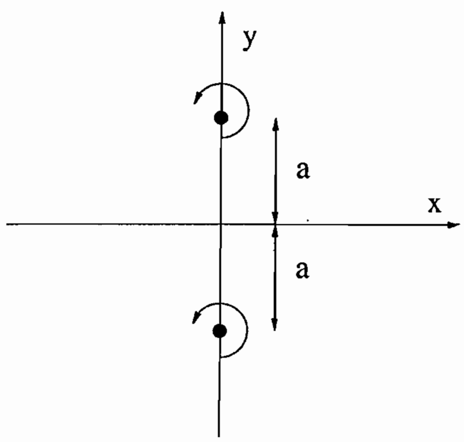


Fig. 1

2 A circular cylinder of radius a in a uniform flow from left to right is to be modelled by a doublet in a uniform flow of speed U.

(a) Write down the complex potential for this flow and find the appropriate strength of the doublet used to model the flow. [20%]

(b) Show that the complex potential you derived satisfies the correct boundary conditions and note the position of the stagnation points. [20%]

(c) A source of strength $2\pi aU$ is added to this flow at $z = -a$ and a sink of strength $-2\pi aU$ at $z = +a$.

(i) How many stagnation points are there now? Indicate roughly where they will be. [20%]

(ii) Find all the stagnation points. Note that the symmetry of the problem will simplify the mathematics. [20%]

(iii) Sketch the flow pattern. [20%]

(TURN OVER

3 A boundary layer developing in a wind-tunnel is found to have a self-similar profile given by

$$\frac{U}{U_\infty} = f(\eta)$$

where $f = 1 - e^{-\eta}$ and $\eta = y/\delta^*$.

(a) Find the momentum thickness, θ, in terms of δ^*, the shape factor H, and the local skin-friction coefficient C_f'. [20%]

(b) Using the results from (a) write down the momentum integral equation for this flow. [20%]

(c) Find an expression for the growth in the displacement thickness with streamwise distance x for the case of zero pressure gradient. [30%]

(d) Find the variation of U_∞ with x that will lead to a constant δ^*. [30%]

4 A long flat plate of negligible thickness is impulsively started from rest such that its velocity (in the direction of its length) increases from zero to U in zero time. In this process a vortex sheet is formed on each side that is initially of zero thickness (and infinite vorticity).

(a) What is the circulation per unit length of the vortex sheet on one side? [20%]

(b) As the plate continues the vortex sheet diffuses according to the equation

$$\omega = \omega_0(t)e^{-y^2/(4\nu t)}$$

where ν is the kinematic viscosity of the fluid, and ω_0 the vorticity, which varies with time, at distance from plate $y = 0$. Use Stokes theorem with this equation to find the circulation per unit length of the vortex sheet. [20%]

(c) By comparing the result in (b) with the circulation per unit length in (a), show that

$$\omega_0 = \frac{U}{\sqrt{\pi \nu t}}$$

Note that: $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. [30%]

(d) Find the drag on the plate per unit length as a function of time – you may assume unit width. [30%]

5 (a) Explain what is meant by irrotational flow and why it is a useful concept. [10%]

(b) Explain physically why an inviscid flow of constant density fluid that is initially irrotational remains irrotational for all time. [10%]

(c) What are the sources of vorticity in homogeneous, constant density flow? [20%]

(d) In a flow with spatially varying density the equation for $D(\omega/\rho)/Dt$ contains the term $\frac{1}{\rho^3}\nabla\rho \times \nabla p$. Explain the meaning of this term and the physical manner in which it can produce vorticity. [20%]

(e) What is vortex stretching? Where might it occur? Use Kelvin's theorem and Stokes theorem to relate the length of a vortex filament to its vorticity in incompressible flow. [20%]

(f) Explain how viscosity limits the intensification of vorticity due to vortex stretching. [20%]

6 (a) Describe the 2D lumped-parameter model of a symmetrical aerofoil section. [20%]

(b) An aerofoil section has camber-line coordinate y_c given by

$$y_c = h\frac{x}{c}\left(1 - \frac{x}{c}\right)^2$$

where h is a constant, c the chord and x the chordwise coordinate. Assuming that thin aerofoil theory applies, calculate [40%]

(i) the lift coefficient at zero incidence;

(ii) the pitching moment coefficient about the quarter-chord point.

(c) A second section, symmetrical and of chord c_t, is placed a distance d downstream of the first. This 'tail' aerofoil is to be set at an angle α_t such that the overall pitching moment about the quarter-chord point of the first ('wing') section is zero. Use the lumped parameter model to estimate α_t. (You may assume that the wing lift and pitching moment are unchanged from (b), and that $d \gg c,\ c_t$.) [40%]

(TURN OVER

7 A straight wing of semi-span s is flying at speed U in a fluid of density ρ. The circulation distribution is

$$\Gamma(y) = \frac{Us}{10}\left[1 - \frac{y^2}{s^2}\right]$$

where y is the spanwise coordinate.

(a) What is the lift force on the wing? [20%]

(b) The circulation distribution may be expressed as the Fourier series

$$\Gamma(y) = Us \sum_{\text{n odd}} G_n \sin n\theta,$$

where θ is defined by $y = -s\cos\theta$. [40%]

(i) Calculate the terms G_1 and G_3. (The standard integrals

$$\int_0^\pi \sin^3\theta d\theta = \frac{4}{3},$$

$$\int_0^\pi \sin^2\theta \sin 3\theta d\theta = -\frac{4}{15}$$

may be assumed without proof.)

(ii) Estimate the percentage by which the wing's drag coefficient exceeds that of one with the same aspect ratio and lift coefficient, but with an elliptical lift distribution.

(c) The wing has a uniform aerofoil cross-section over its span, and chord distribution

$$c(y) = c_r\left(1 - \frac{1}{2}\frac{y}{s}\right),$$

where c_r is a constant. Estimate the position of the maximum section lift coefficient, and comment on the implications for the wing's stall characteristics. [40%]

8 Fig. 2 shows the first return wind tunnel designed by Prandtl in 1907. It turned out that Prandtl was not happy with the flow quality of his design.

(a) Identify the main components included in Fig. 2. Describe briefly what was not good with this wind tunnel. [25%]

(b) How can the design be improved? Illustrate your suggestion with a sketch and explain why this will improve the situation. [25%]

(c) Will the improvement you suggested in (b) have an effect on the wind tunnel's power requirement? Explain your answer. [25%]

(d) One criterion for wind tunnel flow quality is directional uniformity. Give two examples of devices/techniques that can be used to measure flow direction and explain briefly their advantages and drawbacks. [25%]

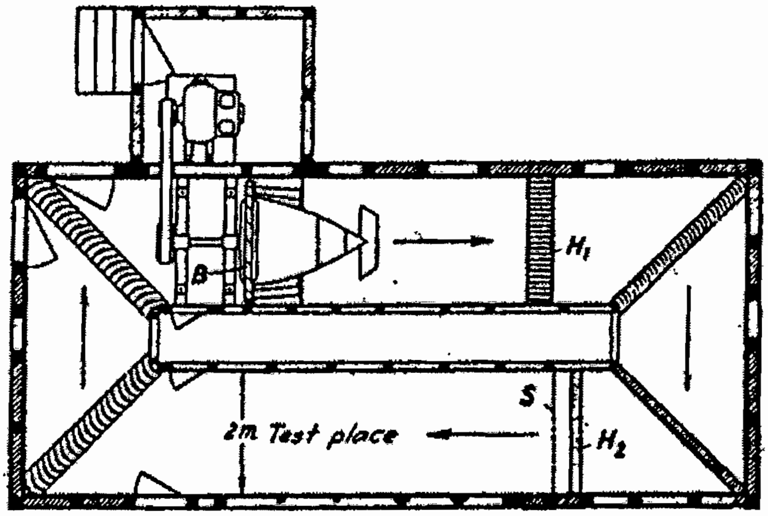


Fig. 2

END OF PAPER

3A1 Data Sheet for Applications to External Flows

Aerodynamic Coefficients

For a flow with free-stream density, ρ, velocity U and pressure p_∞:

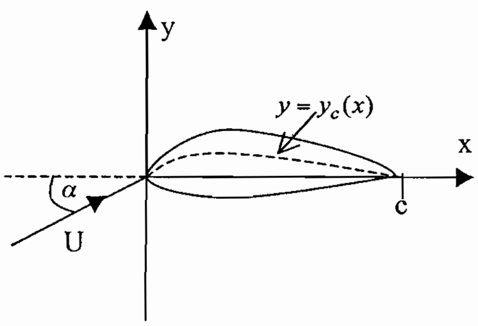
Pressure coefficient: $$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$$

Section lift and drag coefficients: $$c_l = \frac{\text{lift}\,(N/m)}{\frac{1}{2}\rho U^2 c},\quad c_d = \frac{\text{drag}\,(N/m)}{\frac{1}{2}\rho U^2 c}$$ (section chord c)

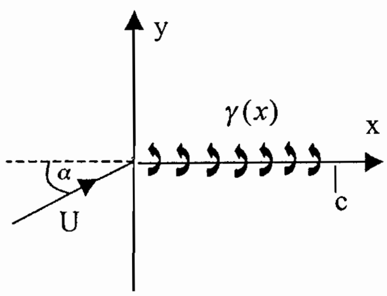
Wing lift and drag coefficients: $$C_L = \frac{\text{lift}\,(N)}{\frac{1}{2}\rho U^2 S},\quad C_D = \frac{\text{drag}\,(N)}{\frac{1}{2}\rho U^2 S}$$ (wing area S)

Thin Aerofoil Theory

Geometry



Approximate representation



Pressure coefficient: $$c_p = \pm\gamma/U$$

Pitching moment coefficient: $$c_m = (\text{moment about } x = 0)/\tfrac{1}{2}\rho U^2 c^2$$

Coordinate transformation: $$x = c(1+\cos\theta)/2 = c\cos^2(\theta/2)$$

Incidence solution: $$\gamma = -2U\alpha\frac{1-\cos\theta}{\sin\theta},\quad c_l = 2\pi\alpha,\quad c_m = c_l/4$$

Camber solution: $$\gamma = -U\left[g_0\frac{1-\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} g_n \sin n\theta\right],$$ where

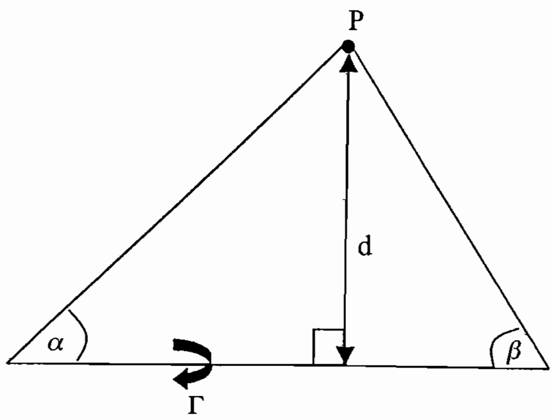
$$g_0 = \frac{1}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)d\theta,\quad g_n = \frac{2}{\pi}\int_0^\pi\left(-2\frac{dy_c}{dx}\right)\cos n\theta\, d\theta$$

$$c_l = \pi\left(g_0 + \frac{g_1}{2}\right),\quad c_m = \frac{\pi}{4}\left(g_0 + g_1 + \frac{g_2}{2}\right) = \frac{c_l}{4} + \frac{\pi}{8}(g_1 + g_2)$$

Glauert Integral

$$\int_{o}^{\pi} \frac{\cos n\phi}{\cos\phi - \cos\theta} d\phi = \pi \frac{\sin n\theta}{\sin\theta}$$

Line Vortices



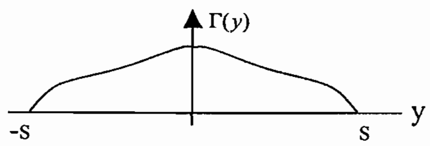
A straight element of circulation Γ induces a velocity at P of

$$\frac{\Gamma}{4\pi d}(\cos\alpha + \cos\beta)$$

perpendicular to the plane containing P and the element.

Lifting-Line Theory

Spanwise circulation distribution:



Aspect ratio: $A_R = 4s^2 / S$

Wing lift: $L = \rho U \int_{-s}^{s} \Gamma(y) dy$

Downwash angle: $\alpha_d(y) = \frac{1}{4\pi U} \int_{-s}^{s} \frac{d\Gamma(\eta)/d\eta}{y - \eta} d\eta$

Induced drag: $D_i = \rho U \int_{-s}^{s} \Gamma(y) \alpha_d(y) dy$

Fourier series for circulation: $\Gamma(y) = Us \sum_{n \text{ odd}} G_n \sin n\theta$, with $y = -s\cos\theta$

Relation between lift and induced drag:

$$C_{Di} = (1 + \delta)\frac{C_L^2}{\pi A_R}, \text{ where } \delta = 3\left(\frac{G_3}{G_1}\right)^2 + 5\left(\frac{G_5}{G_1}\right)^2 + \ldots$$

Module 3A1 – Fluid Mechanics I
Incompressible Flow Data Card

Continuity equation $\nabla \cdot \boldsymbol{u} = 0$

Momentum equation (inviscid) $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

Vorticity $\boldsymbol{\omega} = \text{curl}\, \boldsymbol{u}$

Vorticity equation (inviscid) $\dfrac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}$

Kelvin's circulation theorem (inviscid) $\dfrac{D\Gamma}{Dt} = 0,\ \Gamma = \oint \boldsymbol{u} \cdot d\boldsymbol{l} = \int \boldsymbol{\omega} \cdot d\boldsymbol{S}$

For an irrotational flow

velocity potential (ϕ) $\boldsymbol{u} = \nabla \phi$ and $\nabla^2 \phi = 0$

Bernoulli's equation for inviscid flow,

$$\frac{p}{\rho} + \tfrac{1}{2}V^2 + gz + \frac{\partial \phi}{\partial t} = \text{constant throughout flow field}, \quad V = |\boldsymbol{u}|.$$

TWO-DIMENSIONAL FLOW

Streamfunction (ψ)

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}$$

Lift force Lift / unit length $= \rho U(-\Gamma)$

Complex potential $F(z)$ for irrotational flows, with $z = x + iy$, $F(z) = \phi + i\psi$ and $\dfrac{dF}{dz} = u - iv$

Examples of complex potentials

(i) uniform flow in x direction, $F(z) = Uz$

(ii) source at z_0, $F(z) = \dfrac{m}{2\pi}\ln(z - z_0)$

(iii) doublet at z_0, with axis in x direction, $F(z) = \dfrac{\mu}{2\pi(z - z_0)}$

(iv) anticlockwise vortex at z_0, $F(z) = -\dfrac{i\Gamma}{2\pi}\ln(z - z_0)$

TWO-DIMENSIONAL FLOW

Summary of simple 2 - D flow fields

	ϕ	ψ	circulation	u
Uniform flow (towards $+x$)	Ux	Uy	0	$u = U,\ v = 0$
Source at origin	$\frac{m}{2\pi}\ln r$	$\frac{m}{2\pi}\theta$	0	$u_r = \frac{m}{2\pi r},\ u_\theta = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{2\pi r}$	$-\frac{\mu\sin\theta}{2\pi r}$	0	$u_r = -\frac{\mu\cos\theta}{2\pi r^2},\ u_\theta = -\frac{\mu\sin\theta}{2\pi r^2}$
Anticlockwise vortex at origin	$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi}\ln r$	Γ around origin	$u_r = 0,\ u_\theta = \frac{\Gamma}{2\pi r}$

THREE-DIMENSIONAL FLOW

Summary of simple 3 - D flow fields

	ϕ	u
Source at origin	$-\frac{m}{4\pi r}$	$u_r = \frac{m}{4\pi r^2},\quad u_\theta = 0,\quad u_\phi = 0$
Doublet at origin θ is angle from doublet axis	$\frac{\mu\cos\theta}{4\pi r^2}$	$u_r = -\frac{\mu\cos\theta}{2\pi r^3},\quad u_\theta = -\frac{\mu\sin\theta}{4\pi r^3},\quad u_\phi = 0$

February 2003